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ON LINEAR FAMILIES OF INVOLUTIONS.*

By N. H. Kuiper.

O. Veblen and W. Givens [10, 11] studied the geometry of complex projective spaces. In this paper we follow their method and consider one of their topics: linear families of involutions. In Section **1** we mention the spin representation and we give a classification of (non-singular) linear families of involutions. In Section **2** we introduce an operation Ξ strongly connected with the spin theory. In Section **3**, this is applied to the projective 7-dimensional space. Section **4** deals with a null system in a 6-dimensional quadratic hypersurface, Section **5** with the principle of triality, Section **6** deals with the 15-dimensional projective space. The theory reveals new aspects of the generalizations of the Kummer configurations (Section **7**).

Much of what is presented can be carried over without difficulty to projective spaces over ground fields different from that of the complex numbers.

I wish to thank Professor Oswald Veblen for introducing me to the theory of spinors and for many valuable remarks concerning the content of this paper.

1. Involutions. We shall be concerned with a projective n-dimensional space P_n over the field of complex numbers, and with its linear subspaces $\{P_i\}$, shortly called spaces, of dimension $i = -1, 0, 1, \cdots, n$. After the choice of a coordinate system and homogeneous coordinates in P_n, a point ψ is represented by a class of $(n+1)$-tuples of numbers not all of which vanish ($\rho\psi^A$, $A = 1, \cdots, n+1$; $\rho \neq 0$ variable). The void set P_{-1} is represented by the $(n+1)$-tuple $(0, 0, \cdots, 0)$.

A collineation (γ) is a mapping of the family of subspaces of P_n into itself which preserves intersection relations, and which is determined by the fate of the points $(\psi \to \psi')$, expressed by equations of the form

$$(1) \qquad \rho\psi'^A = \gamma^A{}_B\psi^B; \qquad \text{abbr.: } \rho\psi' = \gamma\psi$$

(the repeated index refers to summation over $B = 1, \cdots, n+1$). The matrices $\gamma^A{}_B$ and $\rho\gamma^A{}_B$ represent the same collineation $(\rho \neq 0)$.

If the square of a collineation γ is the identity mapping, then γ is

* Received June 30, 1949; revised February 16, 1950.

called a non-singular involution and coordinates can be choosen such that the matrix of γ is

$$(2) \qquad \gamma = \left\| \begin{matrix} \underline{1}_p & \\ & -\underline{1}_q \end{matrix} \right\|$$

($\underline{1}_p$ is the unit matrix of p rows and p columns; zero in the empty places; $0 \leqq q, p \leqq n+1 = p+q$). The points invariant under the involution are the points of a P_{p-1} and the points of a P_{q-1}. The involution is called a $(P_{p-1} - P_{q-1})$-reflection. It is the *identity* for $q = 0$ (or $p = 0$).

A collineation γ for which γ^2 transforms any point into P_{-1} is called a singular involution. For suitable coordinates,

$$(3) \qquad \gamma = \left\| \begin{matrix} \underline{0} & \underline{1}r \\ \underline{0} & \underline{0} \end{matrix} \right\|, \quad 2r \leqq n+1.$$

Let $\gamma_1, \cdots, \gamma_{m+1}$ be matrices representing involutions in P_n, and let the same be true for any linear combination $X^1\gamma_1 + X^2\gamma_2 + \cdots + X^{m+1}\gamma_{m+1} \equiv X^\alpha\gamma_\alpha$. Then the set of all these involutions is called a linear family. It follows that

$$(4) \qquad (X^\alpha\gamma_\alpha)^2 = (g_{\alpha\beta}X^\alpha X^\beta) \cdot \underline{1} \qquad (g_{\alpha\beta} = \text{const.}; \alpha, \beta = 1, \cdots, m+1).$$

The involutions of the family are represented by points $(X^1, \cdots, X^{m+1})$ in a projective m-dimensional space R_m. In particular the singular involutions are represented on the quadric $g_{\alpha\beta}X^\alpha X^\beta = 0$. We shall restrict our considerations to non-singular families of involutions, those for which this quadric is not degenerate. A coordinate transformation in R_m corresponds to a change of base for the involutions of the family. For suitable coordinates in R_m we have $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $g_{\alpha\alpha} = 1$. Then the following form is obtained:

$$(5) \qquad (X^\alpha\gamma_\alpha)^2 = (\Sigma_\alpha X^\alpha X^\alpha) \cdot \underline{1}; \qquad \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\delta_{\alpha\beta} \cdot \underline{1}.$$

The space P_n assigned with a non-singular linear family of involutions $L(m+1, n+1)$ is called spin space. Any collineation in P_n which leaves this family of involutions invariant is called a spin transformation. The spin transformations form a group G_s.

The related space R_m is called motion space. A collineation in R_m which leaves the quadric invariant is called a motion. The motions form a group G_m.

It is easily seen that any spin transformation interchanges the involutions of the family in a way which is determined by a motion in R_m, and

any motion in R_m can be obtained in this way. Consequently there is a homomorphism of G_s onto G_m. If this is an isomorphism, then G_s is called the spin representation of G_m.[1] Two non-singular involutions γ_1, γ_2 are called *anti-commuting* if (a) they belong to some pencil of involutions, and (b) they are represented in the related motion space R_1 by points polar with respect to the quadric in R_1. Conditions (a) and (b) are equivalent to the matrix condition

$$\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0. \tag{6}$$

The following normal form is obtained with suitable coordinates in P_n:

$$\gamma_1 = \begin{Vmatrix} & \underline{1} \\ \underline{1} & \end{Vmatrix}, \qquad \gamma_2 = -i \begin{Vmatrix} & \underline{1} \\ -\underline{1} & \end{Vmatrix}; \qquad \text{pencil: } X^1\gamma_1 + X^2\gamma_2. \tag{7}$$

This pencil of involutions introduces another pencil in the line which is the join of the points $(a_1, \cdots, a_r, 0, \cdots, 0)$ and $(0, \cdots, 0, a_1 \cdots, a_r)$, where $2r = n + 1$. The set of such lines is called line regulus. Any line of the line regulus meets any axis $(= P_{r-1})$ of the axis regulus consisting of the one-parameter set of pointwise invariant spaces of the involutions of the pencil. The line and axis regulus cover the same r-dimensional manifold in P_n. If we define an involution of axes on an axis regulus in an obvious way, there results the following geometrical description of a pencil of involutions:

THEOREM 1. *The pairs of invariant spaces of the involutions of a non-singular pencil $L(2, n+1)$ in P_n are the pairs of conjugate axes in an involution of axes on an axis regulus (n is odd by necessity).*

Any non-singular linear family of involutions $L(m+1, n+1)$ $(m \geqq 1)$ admits, for suitable coordinates in P_n and in R_m, the following description (compare (5) and (7)):

$$X^i \begin{Vmatrix} \gamma_i & \\ & -\gamma_i \end{Vmatrix} + X^m \begin{Vmatrix} & \underline{1} \\ \underline{1} & \end{Vmatrix} + X^{m+1}\, i \begin{Vmatrix} & \underline{1} \\ -\underline{1} & \end{Vmatrix}, \tag{8}$$

$$\gamma_i^2 = \underline{1}; \qquad \gamma_i\gamma_j + \gamma_j\gamma_i = 0\,(i \neq j); \qquad i, j = 1, \cdots, m-1.$$

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[1] There is a difference between this theory and the theory which is usually called spin theory, in which a Euclidean centered space E_{m+1} occurs instead of R_m, and an affine centered space A_{n+1} instead of P_n. One first introduces the Euclidean space. Much of our theory can be carried over to this theory in an obvious way. We also remark that we do not enter into interesting problems which occur under restriction of the ground field to that of the real numbers. Cf. references [7, 8, 10, 11, 12, 13, 14, 19].

The last two terms in (8) determine a pencil. Because $X^i\gamma_i$ represents an $L(m-1,(n+1)/2)$, we now easily obtain, by induction on r, the following theorems:

THEOREM 2a. *Non-singular linear families of involutions of the type $L(s, 2^r \cdot p), s > 2r+1$, p odd, do not exist.*

THEOREM 2b. *Any two families of the type $L(2s, 2^r \cdot p), s \leqq r$, are projectively equal.*

THEOREM 2c. *The classes of projectively equal families of the type $L(2s+1, 2^r \cdot p)$, $s \leqq r$, are in one-to-one correspondence with the classes of projectively equal $L(1, 2^{r-s} \cdot p)$ which are involutions in $P(2^{r-s} \cdot p - 1)$. The classification of involutions follows from* (2).

(8) contains the key for the construction of an $L(m+1, 2r)$ from an $L(m-1, r)$. In particular, considering that $L(1,1)$ is the identity transformation in P_0, described by the one-rowed matrix $\gamma_i = \| 1 \|$, $i=1$, we obtain from (8) a normal expression for

$$(9)\qquad \begin{cases} L(3,2): \quad X^1 \begin{Vmatrix} 1 & \\ & -1 \end{Vmatrix} + X^2 \begin{Vmatrix} & 1 \\ 1 & \end{Vmatrix} + X^3 \cdot i \begin{Vmatrix} & 1 \\ -1 & \end{Vmatrix} \\ L(5,4),\ L(2r+1, 2r) \end{cases}$$

(cf. [10], pp. 5-6). For the cases (9) it is known that the spin group G_s is isomorphic to the motion group G_m.

2. The operation Ξ. Let a spin space be given by a non-singular linear family of involutions $L(m+1, n+1)$ in P_n. To every point $\psi \,\varepsilon\, P_n$ we assign a conjugate space $\Xi\psi$, which is the union, and also the join, of all images of ψ under all involutions of $L(m+1, n+1)$. This operation Ξ is invariant under spin transformations, because so is $L(m+1, n+1)$. We shall examine some of these operations which are determined by (maximal) families $L(m+1, n+1) = L(2r+1, 2^r)$.

$L(3,2)$, the first interesting case, consists of all involutions in P_1, except $\gamma = \underline{0}(9)$. Ξ transforms every point ψ onto P_1. Corresponding to every point ψ in P_1, there exists one (singular) involution which transforms ψ onto P_{-1}, and a pencil of involutions which transform ψ into ψ. *Thus points of the spin space* (P_1) *are* 1-1 *represented by points of the invariant quadric* (= *conic section*) *in the motion space R_2, and also* 1-1 *by the tangents to this quadric.* The spin group is obviously the group of all

non-singular projective transformations in P_1, isomorphic with the motion group in R_2.

As to $L(5,4)$, it is seen from (9) that the images under the involutions of $L(5,4)$ of a point with coordinates (a,b,c,d) are the points (x,y,z,u) for which

$$dx - au + bz - cy = 0. \tag{10}$$

Since this is the equation of a null system, we obtain

THEOREM 3a. *The operation Ξ determined by a family of involutions $L(5,4)$ in P_3 is a null system.*

Any line in P_3 has a conjugate line with respect to the null system Ξ (cf. [7]). The self-conjugate lines form a linear line complex. Two pairs of conjugate lines are four lines on a quadric, hence they can be considered as the invariant axes of two involutions of a pencil of involutions (Theorem 1). The family of involutions the invariant spaces of which are the pairs of conjugate lines in the null system, is then a linear family which coincides with $L(5,4)$.

THEOREM 3b. *The pairs of invariant axes of an $L(5,4)$ in P_3 are the pairs of lines conjugate with respect to a null system Ξ in P_3.*

COROLLARY. *The spin group with respect to an $L(5,4)$ is the group of projective transformations which leave a null system invariant (in P_3); it is isomorphic with the group of motions in R_4.*

$L(5,4)$ is a four-dimensional family R_4. Ξ transforms a point into a plane. The linearity of involutions as well as of the family then implies the following fact: Corresponding to every point $\psi \in P_3$, there exists exactly one pencil (R_1) of singular involutions in $L(5,4)$ which transform ψ into P_{-1}, and also one R_2 of involutions all of which transform ψ into ψ. *Hence points of spin space P_3 are 1-1 represented by lines of the three-dimensional invariant quadric in motion space R_4, and also 1-1 by planes (R_2) which are tangent to this quadric along such lines.*

3. The operation Ξ in P_7. The complete description of Ξ determined by a family $L(7,8)$ in P_7 will be given as a summary of the following lemmas.

LEMMA 3.1. *If $\psi \in$ the invariant space of a singular involution of $L(7,8)$, then $\Xi\psi$ is a P_3 and $\psi \in \Xi\psi$.*

Proof. We choose coordinates in motion space and in spin space such that $L(7,8)$ has the expression (9) and that the given singular involution is

$$\tfrac{1}{2}(\gamma_6 - i\gamma_7) = \tfrac{1}{2}\left(\begin{Vmatrix} & \underline{1} \\ \underline{1} & \end{Vmatrix} - i\cdot i \begin{Vmatrix} & \underline{1} \\ -\underline{1} & \end{Vmatrix}\right) = \begin{Vmatrix} & \underline{1} \\ \underline{0} & \end{Vmatrix}.$$

The invariant space a^0 of this involution consists of the points $(a, b, c, d, 0, 0, 0, 0)$ $((\gamma_6 - i\gamma_7)\psi = 0)$. The involutions γ_j, $j = 1, \cdots, 5$, leave a^0 invariant, and they introduce an $L(5,4)$, $X^j\gamma_j$, in a^0. Hence if $\psi = (x^1, x^2, x^3, x^4, 0, 0, 0, 0)$, then $\Xi\psi$ is the join of a plane which contains ψ and is obtained from the null system introduced by that $L(5,4)$ in a^0 (Theorem 3a) *and* the point $\tfrac{1}{2}(\gamma_6 + i\cdot\gamma_7)\psi = (0, 0, 0, 0, x^1, x^2, x^3, x^4)$; q. e. d.

The plane just mentioned consists of the points $(y^1, y^2, y^3, y^4, 0, 0, 0, 0)$ for which

$$x^4y^1 - x^1y^4 + x^2y^3 - x^3y^2 = 0 \tag{11}$$

(cf. (10)). Then $\Xi\psi$ is the set of points $\lambda y^1, \lambda y^2, \lambda y^3, \lambda y^4, \mu x^1, \mu x^2, \mu x^3, \mu x^4)$ for which (11) holds. This implies the following

LEMMA 3.2. *The union of the points $X^\alpha\gamma_\alpha\psi$, ψ variable in a^0, is a quadric Q_6 in P_7, consisting of the points $(y^1, y^2, y^3, y^4, x^1, x^2, x^3, x^4)$ satisfying* (11).

LEMMA 3.3. *If ϕ is a point in Q_6, then $\Xi\phi$ is a P_3 and $\phi \in \Xi\phi$.*

Proof. ϕ is the image under some involution of the family of some point ψ in a^0. The 6-dimensional family of involutions transforms ψ into the points of a P_3. This implies the existence of an R_2 and of an R_3 of involutions in the family all of which transform ψ into P_{-1} and ϕ, respectively. The non-singular involutions of the R_3 also transform ϕ into ψ, and a suitable linear combination of two of these transforms ϕ into P_{-1}. Therefore ϕ is contained in the invariant space of some singular involution of the family, hence Lemma 3.1 can be applied.

LEMMA 3.4 (cf. [17], p. 29). *Q_6 in P_7 contains two families of axes (P_3's), axes^1 and axes^2. The axes of the same (different) family intersect in a P_3, P_1 or P_{-1} (a P_2 or P_0).*

LEMMA 3.5. *The invariant axes of the involutions of $L(7,8)$ are axes of one family (say axes^1) in the Q_6 of Lemma* 3.2.

Proof. $\gamma_6 - i\gamma_7$ determines with an involution $\gamma = X^{1\alpha}\gamma_\alpha$ which does not anti-commute with $\gamma_6 - i\gamma_7$ a non-singular pencil of involutions (γ and $\gamma_6 - i\gamma_7$ are in motion space represented by points not polar with respect to

the invariant quadric in R_6). We may choose coordinates such that this pencil is $X^6\gamma_6 + X^7\gamma_7$ and $L(7,8)$ has the expression (9). Among the images of the points of a^0 under the involutions of the family, we find all points of the line regulus and axis regulus determined by this pencil of involutions (Theorem 1). In particular we find the points of the invariant spaces of γ, which are therefore contained in Q_6. Any involution of the family which *does* anti-commute with $\gamma_6 - i\gamma_7$ is contained in some pencil of involutions all but one of which have invariant space, now known to be contained in Q_6. Hence the invariant axes of any involution of the family are in Q_6. The invariant axes of the involutions of a non-singular pencil are mutually disjoint and therefore (Lemma 3.4) the invariant axes are of one family in Q_6.

LEMMA 3.6. *If $\psi \not\varepsilon Q_6$, then $\Xi\psi$ is the polar hyperplane of ψ with respect to Q_6.*

Proof. ψ is not contained in the invariant space of any singular involution of $L(7,8)$. The images of ψ under the involutions $\gamma_1, \cdots, \gamma_7$ are then seven independent points in P_7, the join of which is a P_6. The points ψ and $\gamma_\alpha\psi$ ($\alpha = 1, \cdot$, or 7) span a line, invariant under γ_α, in which γ_α operates as an (ordinary) involution. Hence ψ and $\gamma_\alpha\psi$ lie harmonic with respect to two points in the invariant spaces of γ_α, that is, in Q_6. Hence Lemma 3.6 follows.

LEMMA 3.7. *If $\psi \varepsilon Q_6$, then $\Xi\psi$ is an axis^2 in Q_6.*

Proof. As in the proof of Lemma 3.3, we find that any point of $\Xi\psi$ is contained in the invariant space of some singular involution of the family and in Q_6. Hence $\Xi\psi$ is an axis in Q_6. For the special choice $\psi \varepsilon a^0$, which is no restriction for the proof, we see that $\Xi\psi$ and the axis^1 a^0 intersect in a plane. This and Lemma 3.4 imply that $\Xi\psi$ is an axis^2.

The correspondence $\psi \rightarrow \Xi\psi$, $\psi \varepsilon Q_6$, can be extended to a 1-1 correspondence of linear spaces in Q_6 for which the name null system in Q_6 will turn out to be appropriate. We summarize the lemmas in

THEOREM 4. *The operation $\psi \rightarrow \Xi\psi$, determined by a family of involutions $L(7,8)$ in P_7, coincides with a polarity with respect to a quadric Q_6 for those points which are not in Q_6; it coincides with a null system in Q_6 for those points which are in Q_6.*

4. The null system Ξ in Q_6.

LEMMA 4.1. *If ψ, ϕ are points in Q_6, and $\phi \varepsilon \Xi\psi$, then $\psi \varepsilon \Xi\phi$.*

It follows, that if l is a line in Q_6, and if $\psi \varepsilon l \subset \Xi\psi$ holds for some point ψ, then the same is true of any point of l. The line l is then self-conjugate with respect to the null system Ξ. The family of all self-conjugate lines is called a linear line complex in Q_6. Obviously the linear line complex in Q_6 determines the null system.

THEOREM 5. *A null system and a linear complex in a Q_6 are determined by an involution of the axes of an axis regulus in that Q_6.*

The figure consisting of the involution of axes[1] which determines the pencil $X^6\gamma_6 + X^7\gamma_7$ in $L(7,8)$ *and* the quadric Q_6 (of Section **3**) has no projective invariants. (Any two such figures are projectively equal.) From the properties of $L(7,8)$ we now obtain the geometrical *construction of the null system Ξ and the line complex,* and hence the proof of Theorem 5, as follows: Fix an involution of axes[1] in an axis[1] regulus in a Q_6. With respect to this involution, three kinds of points (ψ) occur in Q_6, the conjugate spaces of which under the null system are constructed as follows:

(a) ψ is contained in a double axis[1] of the axis[1] involution (say $\psi \varepsilon$ the invariant space a^0 of $\gamma_6 - i\gamma_7$). Then $\Xi\psi$ is the unique axis[2] which contains the regulus line through ψ (the join $\psi \cup \gamma_6\psi$), and which intersects a^0 in a P_2 $(=\psi \cup \gamma_1\psi \cup \gamma_2\psi \cup \gamma_3\psi \cup \gamma_4\psi \cup \gamma_5\psi$; the join).

(b) ψ is contained in another axis[1] of the axis[1] regulus (say in γ_6^+, one of the invariant axes of γ_6). Then $\Xi\psi$ is the unique axis[2] which contains ψ and intersects the conjugate axis[1] (γ_6^-, the other invariant axis of γ_6) in a P_2 $(=\gamma_1\psi \cup \gamma_2\psi \cup \cdots \cup \gamma_5\psi)$.

(c) In the remaining case, let m_1 and m_2 be two lines which contain ψ and each of which intersects two conjugate axis[1] of the axes[1] involution $(m_1 : \psi \cup \gamma_6\psi;\ m_2 : \psi \cup \gamma_7\psi)$. Then $\Xi\psi$ is the unique axis[2], which contains the plane of these two lines (plane: $\psi \cup \gamma_6\psi \cup \gamma_7\psi$).

From now on in this section we shall consider a fixed $L(7,8)$ in the normal form (9). $X^6\gamma_6 + X^7\gamma_7$ is *the* pencil, belonging to *the* line regulus of lines m and to *the* axis[1] regulus, both with *point set M*. It is our aim to extend the null system to a transformation (also called null system) which operates on all points, lines and axes in Q_6.

Corresponding to any axis[2], a^2, we construct a "conjugate" point Ξa_2, of which a^2 is the conjugate. This proves that the mapping $\psi \to \Xi\psi$, $\psi \varepsilon Q_6$,

is a 1-1 mapping of the family of points onto the family of axes^2 in Q_3. First suppose that a^2 intersects one of the axes^1, say a^1, of *the* axis^1 regulus, in a plane (see Lemma 3.4). Then it intersects the others in points. If ψ is one of these points, not in a^1, then a^2 is the unique axis^2 which contains ψ and intersects a^1 in a plane. But then a^2 intersects the other axes^1 of the axis^1 regulus in the points of a regulus line m. And Ξa^2 is the intersection point of m and the axis^1 which is conjugate in the axis^1 involution to a^1, in accordance with (a) and (b) above.

Next, suppose that a^2 intersects all axes^1 of the regulus in points, among others in the points ψ_1 and ψ_2 on the regulus lines m_1 and m_2, respectively. If m_1 equals m_2, we have the case dealt with above. Otherwise the $P_3: m_1 \cup m_2$ intersects M in a two-dimensional quadric Q_2 (= a line regulus), and also in the line $\psi_1 \cup \psi_2$. This is only possible if $m_1 \cup m_2$ is completely contained in Q_6. It intersects the axes^1 of the regulus in lines and is therefore itself an axis^1, say b^1. Then b^1 intersects a^2 among others in the line $\psi_1 \cup \psi_2$, hence in a plane (Lemma 3.4). This plane intersects $Q_2 = b^1 \cap M$ in a conic section. The latter is intersected by the axis^1 involution in a point involution. The joins of corresponding points of this involution are lines of a pencil of lines, the center of which is the point Ξa^2, in accordance with (c).

So far we proved one part of the next theorem, for the formulation of which we need a definition of adjacence: A line and a point are called adjacent if they meet. A line and an axis (in Q_6) are adjacent if the line is contained in the axis. Two other elements of the family of points, lines and axes in Q_6 are adjacent if a third element exists, adjacent with both in the sense defined *so far*. For example, a point and an axis are adjacent if the point is in the axis. Two axes^1 are adjacent if their intersection is a line, etc.

THEOREM 6. *The null system Ξ which operates on points in Q_6 can be extended to a "null system," Ξ, which operates on points, lines axes^1 and axes^2 in Q_6 in the following manner:*

1) *If ψ is a point, then $\Xi\Xi\psi = \psi$.*
2) *If a^2 is an axis^2, then Ξa^2 is a point and $\Xi\Xi a^2 = a^2$.*
3) *If l is a line, so is $\bar{l} = \Xi l$, and $\Xi\Xi l = l$.*
4) *If a^1 is an axis^1, so is $\bar{a}^1 = \Xi a^1$, and $\Xi\Xi a^1 = a^1$.*
5) *The null system preserves the relation of adjacence.*

COROLLARY. *There exists a* 1-1 *rational representation of the axes (say axes^2) of one family on a Q_6, onto the points of Q_6, namely, Ξ.*

Cf. [15]. Intersection with a general P_6 in P_7, which intersects Q_6 in a Q_5, yields a representation of the planes in a Q_5 in terms of the points of a Q_6. Cf. [16], p. 80.

Proof of Theorem 6. The self-conjugate lines were considered immediately after Lemma 4.1. Suppose that l is not a self-conjugate line. Consider two axes^2 which contain l, say a^2 and b^2, with conjugates the points $\psi = \Xi a^2$ and $\phi = \Xi b^2$. If the point $\chi \varepsilon l$, then $\chi \cup \psi$ and $\chi \cup \phi$ are lines of the line complex. Hence the same is true of all lines of the pencil determined by these two lines. Since χ was arbitrary on l, it follows that all lines which meet l and $\bar{l} = \psi \cup \phi$ belong to the linear line complex. The relation between line complex and null system then implies Theorem 6.3 (and part of 6.5).

Next we consider an axis^1, a^1. Any plane in a^1 is contained in exactly one axis^2. The planes of a^1 which contain a line l are contained in axis^2 the conjugates of which are the points of $\bar{l}$. The planes of a^1 which contain at least one edge of a tetrahedron in a^1 are therefore contained in axes^2 the conjugates of which are the points of the edges of another tetrahedron in Q_6. The join of the points of this tetrahedron is an axis in Q_6. Hence, by linearity, the planes of a^1 are contained in axes^2 the conjugates of which are the points of the axis just found. This is not an axis^2, hence an axis^1; so that Ξa^1 is the conjugate, $\bar{a}^1$. Obviously the conjugates of the lines of a^1 are the lines of $\bar{a}^1$. The preservation of adjacence can now be checked for all cases.

The construction of the null system shows that the conjugate of an axis^1 of the axis^1 involution in *the* axis^1 regulus is the conjugate with respect to the null system as well as with respect to the axis^1 involution. The invariant spaces of an involution of *the* pencil $X^6\gamma_6 + X^7\gamma_7$ are conjugate axes^1 of the null system in Q_6. Because the operation Ξ does not distinguish between the non-singular pencils of involutions of $L(7, 8)$, the same is true of the invariant spaces of any involution in $L(7, 8)$.

The converse can be seen as follows: Every line in Q_6 which intersects two different conjugate axes^1 belongs to the linear line complex in Q_6, and every line of this complex which meets an axis^1 also meets its conjugate axis^1. Taking into consideration Theorem 1, it follows that the family of all involutions, having as a pair of invariant spaces a pair of (different) conjugate axes^1 suitably extended with singular involutions, is a linear family; the latter not only contains the given $L(7, 8)$ but coincides with it since it has the same dimension.

THEOREM 7. *The pairs of (possibly identical) invariant spaces of the involutions of a linear family $L(7,8)$ in P_7 are the pairs of axes^1 conjugate with respect to a null system in a quadric Q_6, and vice versa.*

Remarks. Only points in spin space $P_7(L(7,8))$ which are in the Q_6, admit an interesting representation in motion space R_6, in terms of planes in the invariant quadric $Q_5 \subset R_6$ (cf. the end of Section **2**). The spin group consists here of all projective transformations in P_7 which leave quadric and null system invariant, isomorphic with the motion group in R_6.

5. The principle of triality. Consider the family F of linear subspaces, of a dimension number different from 2, of a $Q_6 \subset P_7$: the points (symbol 0), the lines, the axes^1 (symbol 1), and the axes^2 (symbol 2). The triality group $\mathfrak{G}$ of Q_6 is by definition the group of 1-1 mappings of F onto itself, generated by (1) the group G of those collineations in P_7, which leave Q_6 and each family of axes in Q_6 invariant (proper collineations in Q_6); (2) one collineation in P_7 which interchanges axes^1 and axes^2, e. g., a point polar hyperplane reflection Λ in P_7; (3) one null system Ξ in Q_6. All elements of $\mathfrak{G}$ transform lines into lines. $\mathfrak{G}$, considered as a topological space, has the following six components, the elements of which have the indicated activity:

Component		Name of elements	Activity
I:	G	Proper collineations	$0 \to 0, 1 \to 1, 2 \to 2$
II:	$G\Lambda$	Improper collineations	$0 \to 0, 1 \rightleftarrows 2$
III:	$G\Xi$	^{1}correlations	$1 \to 1, 0 \rightleftarrows 2$
IV:	$G\Lambda\Xi\Lambda$	^{2}correlations	$2 \to 2, 0 \rightleftarrows 1$
V:	$G\Lambda\Xi$	^{1}trialities	$0 \to 1 \to 2 \to 0$
VI:	$G\Xi\Lambda$	^{2}trialities	$0 \leftarrow 1 \leftarrow 2 \leftarrow 0$

The extended triality group in Q_6 is generated by $\mathfrak{G}$ and one anti-involution in P_7 which leaves the quadric invariant. If the coefficients of the equation which represents Q_6 are real (as above), then such an anti-involution is obtained from the automorphism $i \to -i$ of the ground field. The anti-involution then consists in the mapping of any point onto the point with complex conjugate coordinates. This makes obvious the following fact (cf. [1, 2]):

THEOREM 8. *The transformations of the (extended) triality group preserve the relations of adjacence between points, lines and axes in Q_6.*

This theorem is the expression for a principle of triality analogous to the principle of duality (point-hyperplane, etc.) in the geometry of the projective space P_n. By the principle of triality, theorems on points, lines, and axes in Q_6 can be carried over into other theorems. Consider, for example, the following theorem (cf. [18], p. 51): A 1-1 point transformation of Q_6, which preserves adjacence of points, is a proper or improper collineation or anti-collineation of Q_6 (if Q_6 is represented by an equation with real coefficients, then the "anti" refers to the automorphism $i \to -i$). The triality principle transforms this theorem into

THEOREM 9. *A 1-1 adjacence-preserving transformation of the axes*[1] *in a Q_6 is a proper collineation, a proper anti-collineation, a* ^{1}*correlation or an anti-*^{1}*correlation (an example is the null system).*[2]

Another application of the principle of triality results as follows: Under a ^{2}correlation the improper collineations in Q_6 are interchanged with the ^{1}correlations; in particular the point-hyperplane reflections of Q_6 (in P_7) are interchanged with the ^{1}null systems (like Ξ). The ^{1}null systems as well as the $L(7,8)$'s with a fixed Q_6 in P_7, are therefore rationally representable (1-1) by the points of a P_7 not in a Q_6 in that P_7. The points of the Q_6 correspond to degenerated null systems.

6. The operation Ξ in P_{15}. In this section Ξ will indicate the operation which assigns, to every point $\psi \in P_{15}$, the linear space which is the join of the images of ψ under the involutions of an $L(9,16)$. The analysis of Ξ is analogous to that in preceding cases. Let $L(9,16)$ have the normal form (9). Then $X^8\gamma_8 + X^9\gamma_9$ is *the* pencil of involutions, the invariant spaces of which form *the* axis regulus (P_7's) with M as point set. M also covers *the* line regulus (Theorem 1). The invariant spaces of $\gamma_8 - i\gamma_9$ and $\gamma_8 + i\gamma_9$

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[2] *Remarks*: Wei Liang Chow ([18] p. 55) proved that an adjacence-preserving 1-1 mapping of the axes (P_r) of one family in a quadric Q_{2r} is a collineation or an anti-collineation if $r > 3$. He drew my attention to the interesting exceptional case $r = 3$, of which Theorem 9 gives a specific description.

The principle of triality originates from E. Study [6]. In his work he gives an interesting interpretation of the principle of triality in terms of point pairs, proper motions and improper motions in an elliptic three-dimensional space.

Many applications of the principle are found in the book of Weiss ([16], p. 154 ff.).

E. Cartan [7] arrived at the principle of triality in group theory. Considering Lie groups and their automorphisms, he has shown that the group G^* of collineations and anti-collineations in Q_6 is exceptional. All its automorphisms are obtained from the inner automorphisms ($\mathfrak{a} \to \mathfrak{b}^{-1}.\mathfrak{a}.\mathfrak{b}$) of the extended triality group. The factor group of the group of automorphisms over the group of inner automorphisms of G^* is of order 6.

are a^0 and b^0, respectively. $X^i\gamma_i$, $i=1,\cdots,7$, introduces an $L(7,8)$ in the P_7 a^0. The invariant spaces of these involutions in a^0 lie in the Q_6 of points $(x^1,x^2,x^3,x^4,y^1,y^2,y^3,y^4,0,\cdots,0)$ obeying (11) of Section **3**.

If ψ $(x^1,\cdots,y^4,0,\cdots,0)$ is in a^0 but not in Q_6, then $\Xi\psi$ is a P_7 which does not contain ψ and which is the join of the polar-P_6 of ψ with respect to the Q_6 in a^0, and the point $\gamma_8\psi=-\gamma_9\psi$ $(0,\cdots,0,x^1,\cdots,y^4)$ in b^0. For ψ variable (in a^0 but not in Q_6), the spaces $\Xi\psi$ cover the set of points $(u^1,u^2,u^3,u^4,v^1,\cdots,v^4,x^1,\cdots,x^4,y^1,\cdots,y^4)$

$$x^4v^1+u^4y^1-x^1v^4-u^1y^4+x^2v^3+u^2y^3-x^3v^2-u^3y^2=0, \tag{12}$$

$$x^4y^1-x^1y^4+x^2y^3-x^3y^2\neq 0 \tag{13}$$

(cf. (11)). Any point of this set is contained in the invariant space of some singular involution of $L(9,16)$ (cf. the end of Section **2**). Conversely, if there is given in the family some singular involution which does not anti-commute with $\gamma_8-i\gamma_9$, then its invariant space is contained in the Q_{14} of (12). This is seen from the example $\gamma_8+i\gamma_9$. The locus of points contained in the invariant space of some singular involution of $L(9,16)$ is an algebraic manifold, hence it is (12). If two singular involutions do not anti-commute, then they are two involutions of a non-singular pencil, and their invariant spaces are disjoint (cf. the representing points in motion space R_8). In consequence of the generalization of Lemma 3.4, these invariant spaces are axes of one family in the Q_{14}, (12). Hence the invariant axes of the singular involutions of $L(9,16)$ are (say) axes[1] in Q_{14}.

If ψ is in a^0 but not in Q_6 (in a^0), then $\Xi\psi$ is a P_7 in Q_{14} and intersects a^0 in a P_6. Then $\Xi\psi$ is an axis[2] of Q_{14}. The same is true of any point $\psi\,\varepsilon\, Q_{14}$ for which $\Xi\psi$ is a P_7. All these points lie in the invariant space of exactly one singular involution of the family. Because $\psi\not\varepsilon\,\Xi\psi$, these points are not contained in the invariant space of any non-singular involution of $L(9,16)$.

Because every point in Q_{14} lies in the invariant space of some singular involution, all kinds of points can be found in a^0. So we still have to consider the points in the Q_6 in a^0. If ϕ is such a point, then $\Xi\phi$ is the P_4 spanned by the $P_3: \phi\cup\gamma_1\phi\cup\cdots\cup\gamma_7\phi$ and the point $\gamma_8\phi$ in b^0. The locus of all points like ϕ in Q_{14} has at the point $\gamma_8\phi$ a tangent space, which contains the P_4 just mentioned and also tangent P_6 to *the* Q_6 in b^0. This locus is therefore of dimension $\geqq 10$. It is the intersection of the cone-like manifolds C_{14},[3] which are constructed as follows: Take any two not anti-commuting

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[3] T. G. Room has pointed out that this locus is an intersection of quadratic hypersurfaces C_{14}.

singular involutions of $L(9,16)$, e. g., $\gamma_8 - i\gamma_9$ and $\gamma_8 + i\gamma_9$ with invariant spaces a^0 and b^0. The locus of the joins of one point in b^0 and one point in the invariant Q_6 in a^0 is a C_{14} as required:

$$(u^1,\cdots,v^4,x^1,\cdots,y^4) \qquad x^4y^1 - x^1y^4 - x^2y^3 - x^3y^2 = 0.$$

The intersection of all C_{14}'s is easily seen to be of dimension $\leqq 10$, hence of dimension $= 10$. We call it T_{10}.

The algebraic manifold which consists of all points of P_{15} contained in the invariant space of some involution of $L(9,16)$, contains Q_{14} but also *the* axis regulus of $X^8\gamma_8 + X^9\gamma_9$. This locus is therefore of dimension > 14, and must be P_{15} itself.

Apart from some details we have now proved

THEOREM 10. *A non-singular family of involutions $L = L(9,16)$ in P_{15} determines an invariant quadric Q_{14}, and an invariant manifold $T_{10} \subset Q_{14}$ which is an intersection of cones C_{14}. Every invariant axis of every involution of L meets T_{10} in a Q_6. Every point ψ in P_{15}, but not in T_{10}, is contained in an invariant axis of exactly one involution of L, which is singular (non-singular) if ψ is contained (not contained) in Q_{14}. The invariant axes of the singular involutions of L are axes of one family in Q_{14} (say axes*[1]*).*

The operation Ξ determined by $L(9,16)$ is as follows: (1) *If $\psi \not\varepsilon Q_{14}$, then $\Xi\psi$ is a P_8 which contains ψ and intersects T_{10} in a Q_6.* (2) *If $\psi \varepsilon Q_{14}$, $\psi \not\varepsilon T_{10}$, then $\Xi\psi$ is an axis*[2] *in Q_{14} which does not contain ψ and intersects T_{10} in a Q_5.* (3) *If $\psi \varepsilon T_{10}$, then $\Xi\psi$ is a P_4 in T_{10}, and $\psi \varepsilon \Xi\psi$.*

Concerning the general case of an operation Ξ determined by a linear family of involutions $L(2r+1, 2^r)$ in P_{2^r-1}, we only mention the following property: Let ψ be a point in this space and let $r - \lambda(\psi)$ be the maximal number of linearly independent, singular involutions of the family with invariant spaces which contain ψ. Then either $\lambda = 0$ or $3 \leqq \lambda \leqq r$. If $\lambda = 0, 4$, then $\psi \varepsilon \Xi\psi$. If $\lambda = 3$, then $\psi \not\varepsilon \Xi\psi$. Otherwise both cases may occur. If $\lambda = 0$, then the point ψ in spin space is represented (1-1) by an R_{r-1} in motion space R_{2r}.

O. Veblen and W. Givens [10] has shown that an $L(2r+1, 2^r)$ determines an invariant involutory correlation C, which is a polarity in a quadric if $r \equiv 0, 3 \pmod 4$ and a null system (in the ordinary sense) if $r \equiv 1, 2 \pmod 4$. If $r = 1, 2$, then C coincides with our operation Ξ. If $r = 3, 4$, then C is the polarity with respect to the invariant quadric (Q_6, Q_{14}) which we mentioned. If $r = 3$, then C coincides nearly with Ξ. For $r \geqq 4$, C and Ξ are no longer related in such a simple way.

7. Kummer configurations. It was suggested by T. G. Room that there might exist connections between the subject of this paper and the configuration $Cf(16_6)$ of Kummer [1, 3, 4], generalized by Wirtinger [2], Barrau [5] and Room [9]. We shall show that such connections exist, and that the operation Ξ reveals some new aspects of those configurations.

The basic figure is a set of $2r+1$ anti-commuting involutions $\gamma_1, \cdots, \gamma_{2r+1}$ in P and the invariant involutory correlation C in P_{2r-1} (cf. above). The involutions are the base of a linear family $L(2r+1, 2^r)$ which again determines an operation Ξ. The involutions $\gamma_1, \cdots, \gamma_{2r+1}$ and the correlation C generate a group F consisting of 2^{2r} involutions and 2^{2r} involutoric correlations of which $(2^r-1)2^{r-1}$ are null systems and $(2^r+1)2^{r-1}$ are polarities in quadrics (cf. [5]). The images of one point ψ in P_{2^r-1} under F are 2^r points and 2^r hyperplanes, between which strong incidence relations hold. Every hyperplane of the set contains (at least) $(2^r-1)2^{r-1}$ points of the set, and every point is contained in (at least) equally many hyperplanes of the set (because F contains that same number of null systems).

If ψ is specially chosen in one of the quadratic hypersurfaces, then the tangent hyperplane at ψ to this hypersurface belongs to the set of points and hyperplanes, and it contains in this case at least $(2^r-1)2^{r-1}+1$ points. F operates transitively on the set of points and hyperplanes, and F preserves incidence relations, hence the same (the dual) is then true of all hyperplanes (points) of the set.

In P_3 we obtain in this way (choice of ψ not special) the configuration of Kummer [9, 11, 13], consisting of 16 points and 16 planes. Each plane contains 6 points and each point is contained in 6 planes. This situation is obvious from our considerations, since if $\psi \varepsilon P_3$, then $\psi, \gamma_1\psi, \cdots, \gamma_5\psi$ lie in the plane $\Xi\psi$.

A special configuration is obtained if ψ lies on a line which intersects the invariant axes (= lines) of (say) γ_1 and γ_2. Then the points $\psi, \gamma_1\psi, \gamma_2\psi, \gamma_1\gamma_2\psi$ lie on one line, and all 16 points lie on a set of four lines.

In P_7 we obtain the configuration of Barrau. If $\gamma_1, \cdots, \gamma_7$ are constructed as in (9) of Section **1**, then the points will have the same coordinates as those obtained from the device of Barrau. The configuration $Cf(64_{28})$ consists of 64 points and 64 hyperplanes in P_7, each of which is incident with (at least) 28 of the other kind. The 36 polarities in F determine 36 Q_6's in P_7. The Q_6 determined by $L(7, 8)$ is a representative of these.

Suppose now that we choose the point ψ in this Q_6. The involutions $\gamma_1, \cdots, \gamma_7$, and all other involutions of F, transform ψ into the Q_6; so that the points $\psi, \gamma_1\psi, \cdots, \gamma_7\psi$ are contained in the axis[1] $\Xi\psi$ in Q_6 (Section **3**).

Because F operates transitively on the set of points and hyperplanes, the following theorem holds:

THEOREM 11. *If ψ is contained in the invariant Q_6 of one of the 36 polarities, elements of the group F of correlations and involutions generated by 7 anti-commuting involutions and the related invariant polarity in P_7, then the images of ψ under F form a configuration $Cf(64, 29, 8)$.*

This configuration consists of 64 points, 64 hyperplanes and 64 solids (P_3's). Every hyperplane contains (at least) 29 points and (at least) 8 solids. Every point is contained in (at least) 29 hyperplanes and 8 solids. Every solid contains (at least) 8 points and is contained in (at least) 8 hyperplanes.

Room obtained this configuration in a different way [9]. Theorem 11 could also be stated as follows: The configuration of Barrau, which is special in the sense indicated above, has all incidence properties known of the configuration of Room.

In the generalization to higher dimensions, the operation Ξ determined by a linear family of involutions $L(2r+1, 2^r)$ distinguishes between different kinds of points ψ, and this distinction carries with itself a distinction between different kinds of more or less specialized generalized Kummer configurations.

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ON THE EXISTENCE OF LAPLACE SOLUTIONS FOR LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER.*

By Aurel Wintner.

1. Let $\alpha(t)$, where

$$(1) \qquad 1 \leqq t < \infty,$$

be a (possibly complex-valued) function of finite total variation

$$(2) \qquad [\alpha] = \int_1^\infty |\, d\alpha(t)|,$$

and let $f(s)$, where

$$(3) \qquad 0 \leqq s < \infty,$$

denote the function

$$(4) \qquad f(s) = \int_1^\infty e^{-st} d\alpha(t), \text{ where } [\alpha] < \infty.$$

The latter will be chosen to be the coefficient function of the differential equation

$$(5) \qquad x'' + f(s)x = 0, \qquad (' = d/ds),$$

about which the following theorem will be proved:

(i) *If $f(s)$ is representable, on the half-line* (3), *in the form* (4), *then* (5) *has a solution $x(s) \not\equiv 0$ representable, on the same half-line, in the form*

$$(6) \qquad x(s) = \text{const.} + \int_1^\infty e^{-st} d\beta(t), \text{ where } [\beta] < \infty,$$

with the const. *as an arbitrary "initial value" $x(\infty)$,*

$$(7) \qquad x(s) \to \text{const. as } s \to \infty,$$

where const. $= 0$ *only when* $x(s) \equiv 0$.

In some respect, (i) is related to a theorem proved in [2] for systems of first order, say

$$(8) \qquad {}^i x' = \sum_{k=1}^n f_{ik}(s)\, {}^k x, \qquad (i = 1, \cdots, n),$$

* Received July 3, 1949.

where every $f_{ik}(s)$ is of the form (5). But (5) presents an entirely different problem. In fact, if (5) is written in the form (8), as

$$x' = 0 \cdot x + 1 \cdot y, \qquad y' = -f(t)x + 0 \cdot y \tag{8 bis}$$

($^1x = x$, $^2x = y$; $n = 2$), then the condition imposed on the n^2 coefficient function $f_{ik}(s)$ of (8) is not satisfied by $f_{12}(s) \equiv 1$, since the latter function results by choosing $\alpha(t) = \operatorname{sgn} t$ but replacing (1) by $0 \leqq t < \infty$ in (4). Trivial examples show, however, that the results of [2] cannot apply if the range of integration reaches down to $t = 0$.

Actually, even the assertion of the theorem on (8) becomes quite different. In fact, the result of [2] is that the components $^ix(s)$ of every solution of (8) are of the form (6), whereas (i*) below will show that such is never the case for (5). Thus the assertion of (i) is closer to the result of [1] than to that of [2]. In fact, [1] deals with (5), but it assumes that $d\alpha(t) \leqq 0$ in (4), whilst it allows that (1) in (4) be replaced by $0 \leqq t < \infty$.

2. The fact referred to above is as follows:

(i*) *Under the assumptions of* (i), *the solution supplied by* (i) *is unique to a constant factor.*

In other words, (5) cannot have two, linearly independent, solutions of the form (6). This can be seen as follows: It is clear from (6) that $x(s) = O(1)$ and $x'(s) = o(1)$, as $s \to \infty$. Hence, if $x = u(s)$ and $x = v(s)$ are two solutions of the form (6), then their Wronskian, $u(s)v'(s) - v(s)u'(s)$, is $o(1)$. But the Wronskian of two linearly independent solutions of (5) is a non-vanishing constant and cannot, therefore, be $o(1)$. This proves (i*).

If $f(s)$ is of the form (4), let $S(\alpha)$, the "spectrum" of α, be defined as the set of those t-values which are points of non-constancy of the function $\alpha(t)$. Let $S^*(\alpha)$ denote the closure of the set of those t-values which are representable in the form $n_1t_1 + \cdots + n_mt_m$, where m and $n_1, \cdots, n_m$ denote positive integers and $t_1, \cdots, t_m$ are points of $S(\alpha)$. Then the assertion of (i) can be completed as follows:

(i bis) *Under the assumptions of* (i), (i*), *the spectrum* $S(\beta)$ *defined by* (6) *is contained in the set* $S^*(\alpha)$.

This will be clear from the proof of (i) below, if recourse is had to the "addition rule of spectra" in case of convolutions.

3. The proof of (i) depends on a process of successive approximations. Formally, the successive approximations, $x_0(s), \cdots, x_n(s), \cdots$, to the solution in question can be introduced as follows:

$$x_{n+1}(s) = 1 + \int_s^\infty (s-t)f(t)x_n(t)\,dt, \tag{9}$$

where $0 \leqq s < \infty$ and

$$x_0(s) \equiv 0. \tag{10}$$

If

$$x_{n+1}(s) = 1 + \sum_{m=1}^{n} y_m(s), \tag{11}$$

then (9) and (10) are equivalent to

$$y_{n+1}(s) = \int_s^\infty (s-t)f(t)y_n(t)\,dt, \tag{12}$$

where

$$y_0(s) \equiv 1. \tag{13}$$

It will first be shown that this recursion rule leads to a representation

$$y_n(s) = \int_1^\infty e^{-st}d\gamma_n(t), \qquad (n > 0), \tag{14}$$

where $\gamma_n(t)$ is a function satisfying $[\gamma_n] < \infty$, if the bracket is defined by (2).

According to (13) and the case $n = 0$ of (12),

$$y_1(s) = \int_s^\infty (s-t)f(t)\,dt. \tag{15}$$

If f is substituted from (4) into (15), and if the order of resulting integrations is interchanged (which, in view of $[\alpha] < \infty$, is legitimate, by absolute convergence), then it is seen from the identity

$$\int_s^\infty (s-t)e^{-tu}dt = -e^{-us}/u^2, \text{ where } 0 < u < \infty, \tag{16}$$

that (15) can be written as $y_1(s) = -\int_1^\infty e^{-st}t^{-2}d\alpha(t)$. Hence, (14) holds for $n = 1$, with

$$d\gamma_1(t) = -t^{-2}d\alpha(t). \tag{17}$$

In order to carry out the induction from n to $n+1$, suppose that (14) holds for a fixed n and for some $\gamma_n(t)$ satisfying $[\gamma_n] < \infty$. Then, by (4),

$$f(s)y_n(s) = \int_1^\infty e^{-st}d(\alpha * \gamma_n)(t), \tag{18}$$

where the asterisk denotes the operation of convolution, that is,

$$(\lambda * \mu)(t) = \int_1^t \lambda(t-u)d\mu(u) = (\mu * \lambda)(t),$$

if $[\lambda] < \infty$ and $[\mu] < \infty$. But if fy_n is substituted from (18) into (12), and if (16) is then used in the same way as above, it is seen that

$$y_{n+1}(s) = -\int_1^\infty e^{-st}t^{-2}d(\alpha * \gamma_n)(t).$$

Accordingly, n in (14) can be replaced by $n+1$ if $\gamma_{n+1}(t)$ is defined, corresponding to (17), by

$$d\gamma_{n+1}(t) = -t^{-2}d(\alpha * \gamma_n)(t). \tag{19}$$

This completes the induction.

4. Needless to say, γ_n can be left undetermined at its points of discontinuity and to an additive constant, since $[\gamma_n] < \infty$. For instance, it can be assumed that

$$\gamma_n(1) = 0, \text{ and } \gamma_n(t-0) = \gamma_n(t) \text{ if } 1 < t < \infty. \tag{20}$$

It turns out that $\gamma_n(t)$ is independent of t if $1 \leqq t < n$ or, in view of the first of the relations (20), that

$$\gamma_n(t) = 0 \text{ if } 1 \leqq t < n, \text{ where } n > 1;\ \gamma_1(1) = 0. \tag{21}$$

This follows from (17) and (19), where $1 \leqq t < \infty$, by an induction from n to $n+1$, if use is made of the "addition rule of spectra." In fact, this induction is exactly the same as that given in [2], pp. 335-336, and will therefore be omitted.

On the other hand, it is readily seen from the definitions of the symbols

[ ], * that $[\lambda * \mu] \leqq [\lambda][\mu]$. It follows therefore from (19) and (21) that $[\gamma_{n+1}] \leqq (n+1)^{-2}[\alpha][\gamma_n]$. This recursive inequality, when applied to $n = 1, 2, \cdots$, clearly implies that $[\gamma_n] = O(a^n/n!^2)$ as $n \to \infty$, where a is independent of n. In particular

$$\sum_{n=1}^{\infty} [\gamma_n] < \infty. \tag{22}$$

It is clear from (22) and from the first of the relations (20) that the series

$$\beta(t) = \sum_{n=1}^{\infty} \gamma_n(t), \text{ where } 1 \leqq t < \infty, \tag{23}$$

defines a function satisfying $\beta(1) = 0$ and $[\beta] < \infty$. It is also seen that the function (23) is such as to satisfy, for $0 \leqq s < \infty$, the limit relation

$$\sum_{m=1}^{n} \int_1^{\infty} e^{-st} d\gamma_m(t) \to \int_1^{\infty} e^{-st} d\beta(t), \text{ as } n \to \infty.$$

In view of (14) and (11), this means that

$$x_n(s) \to x(s) \text{ as } n \to \infty;\ 0 \leqq s < \infty, \tag{24}$$

if $x(s)$ is defined by

$$x(s) = 1 + \int_1^{\infty} e^{-st} d\beta(t) \qquad ([\beta] < \infty). \tag{25}$$

Finally, it is readily seen from (22), (23) and (4) that the limit process (24) can be carried out beneath the integral sign on the right of (9). This means that the function (25) is a solution of the integral equation

$$x(s) = 1 + \int_s^{\infty} (s-t) f(t) x(t)\, dt; \qquad 0 \leqq s < \infty. \tag{26}$$

Two differentiations of (26) show that the function $x(s)$ is a solution of (5). In addition, it is clear from (25) that $x(s)$ satisfies the case const. $= 1$ of (7), (6). Since a solution of (5) can be multiplied by an arbitrary constant, it follows that, in order to complete the proof of (i), only the last assertion of (i), that following (7), remains to be ascertained. But the truth of this assertion is contained in the uniqueness statement of (i*), which was verified above.

5. Nothing is changed in the above proofs if the half-line (3) is replaced by the half-plane $\sigma \geqq 0$, where $s = \sigma + it$, $-\infty < t < \infty$. In particular, if the claims of (i), (i*), (i bis) are applied on the boundary of the half-plane $\sigma \geqq 0$, there results the following theorem:

(ii) *Let a continuous $f(t)$, where $-\infty < t < \infty$, be a function representable in the form*

$$(27) \qquad f(t) = \int_1^\infty e^{its} d\alpha(s), \text{ where } \int_1^\infty |\, d\alpha(s) \,| < \infty.$$

Then, corresponding to every constant c, the differential equation

$$(28) \qquad x'' + f(t)x = 0 \qquad (' = d/dt)$$

has, for $-\infty < t < \infty$, a unique solution of the form

$$(29) \qquad x(t) = c + \int_1^\infty e^{its} d\beta(s), \text{ where } \int_1^\infty |\, d\beta(s) \,| < \infty.$$

The spectrum $S(\beta)$ is contained in the set $S^*(\alpha)$.

Obviously, the lower limit, 1, of the integrations (27), (29) can be replaced by any positive number.

Of particular interest is the case in which the function α is a step-function. Then (ii) leads to the following theorem:

(iii) *Let $f(t)$ be a uniformly almost-periodic function of the form*

$$(30) \qquad f(t) \sim \sum_{n=1}^\infty a_n e^{i\lambda_n t}, \text{ where } \lambda_n > \text{const.} > 0,$$

and suppose that

$$(31) \qquad \sum_{n=1}^\infty |\, a_n \,| < \infty.$$

Then the differential equation (28) *has a uniformly almost-periodic solution of the form*

$$(32) \qquad x(t) \sim c + \sum_{n=1}^\infty b_n e^{i\mu_n t}, \qquad (\mu_n > \text{const.} > 0),$$

where every exponent μ_n is a linear combination, with positive integral coefficients, of the exponents λ_n, and

$$(33) \qquad \sum_{n=1}^\infty |\, b_n \,| < \infty.$$

Furthermore, the assignment of an arbitrary mean-value

$$c = \lim_{v-u\to\infty} (v-u)^{-1} \int_u^v x(t)\,dt$$

as an integration constant determines the solution (32) *uniquely.*

6. In view of the uniqueness theorem expressed by (i*), and of the corresponding assertion of (ii), there arises the question as to the form of another (linearly independent) solution or, equivalently, of the general solution of the differential equations (5), (28) under the respective assumptions of (i), (ii). It will be sufficient to develop the answer to this question for the case of (i), since it then follows by the above device, $s = it$, for the cases of (ii) and (iii). In the case of (i), the theorem is as follows:

(I) *If* $f(s)$ *is representable, on the half-line* (3), *in the form* (4), *then the general solution of* (5) *is* $x(s) = c_1x_1(s) + c_2x_2(s)$, *where* c_1, c_2 *are arbitrary constants, and* $x_1(s)$, $x_2(s)$ *solutions representable, on the half-line* (3), *as*

$$(34) \qquad x_1(s) = 1 + \int_1^\infty e^{-st}d\beta(t)\,;\ [\beta] < \infty, \text{ and } x_2(s) = sx_1(s) + y(s),$$

respectively, where $y(s)$ *is a function of the form*

$$(35) \qquad y(s) = \int_1^\infty e^{-st}d\gamma(t)\,; \qquad [\gamma] < \infty.$$

First, $x_1(s)$ in (34) is the solution (25), supplied by (i). On the other hand, substitution of $x_2(s)$ from (34) into (5) gives

$$(36) \qquad y'' + f(s)y = g(s),$$

where

$$(37) \qquad g(s) = -2x_1'(s),$$

since

$$(38) \qquad x_1''(s) = -f(s)x_1(s),$$

by (5). Hence, in order to prove (I), it is sufficient to show that the inhomogeneous differential equation (36) has, in the case (37), a solution of the

form (35). In fact, since the definite integrals occurring in (34) and (35) tend to 0 as $s \to \infty$, it is clear from (34) that $x_1(s) \to 1$ but $|x_2(s)| \to \infty$, which assures that $x_2(s)$ is linearly independent of $x_1(s)$.

In order to prove for (36) the existence of a solution (35), use will be made of the fact that the function (37) is representable in the form

$$(39) \qquad g(s) = \int_1^\infty e^{-st} d\phi(t); \qquad [\phi] < \infty.$$

This can be verified as follows:

By (4) and the first of the relations (35),

$$f(s)x_1(s) = \int_1^\infty e^{-st} d\alpha(t) + \int_1^\infty e^{-st} d\alpha*\beta(t).$$

On the other hand, from (38) and the first of the relations (35),

$$\int_1^\infty e^{-st} t^2 d\beta(t) = -f(s)x_1(s).$$

Consequently, $t^2 d\beta(t) = -d\alpha(t) - d\alpha*\beta(t)$. In view of $[\alpha*\beta] \leqq [\alpha][\beta]$, where $[\alpha] < \infty$ and $[\beta] < \infty$, this implies, by (1), that

$$\int_1^\infty t^2 \,|\, d\beta(t) \,| < \infty; \text{ in particular, } \int_1^\infty |\, t d\beta(t) \,| < \infty.$$

It follow therefore from (37) and from the first of the relations (34) that (39) is satisfied by $d\phi(t) = -2t d\beta(t)$.

Accordingly, more than what is needed for (I) is contained in the following theorem:

(II) *If $f(s)$, $g(s)$ are functions representable, on the half-line* (3), *as integrals of the form* (4), (39), *then the inhomogeneous differential equation* (36) *has, on the same half-line, a solution of the form* (35).

The proof of (II) proceeds again by successive approximations, except that (9) must be replaced by

$$y_{n+1}(s) = h(s) + \int_1^\infty (s-t) f(t) y_n(t) dt, \text{ where } h(s) = \int_1^\infty e^{-st} t^{-2} d\gamma(t),$$

and (10) by $y_0(s) \equiv 0$ or $y_1(s) = h(s)$. Due to the assumption (34), the induction from n to $n+1$ and the limit process $n \rightarrow \infty$ can then be carried out in exactly the same way as in the proof of (i), and will not therefore be repeated.

Clearly, both (I) and (II) can be replaced by the theorems which relate to (I) and (II) in the same way as either (ii) or (iii) related to (i).

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ISOMORPHIC GROUPS OF LINEAR TRANSFORMATIONS.* [1]

By C. E. Rickart.

Introduction. Consider a system $\{\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^* : \mathfrak{G}\}$ of the following type: $\mathfrak{X}$, $\mathfrak{X}^*$ are dual [3, p. 15] (right and left respectively) linear vector spaces over the division ring $\mathfrak{D}$ (not of characteristic 2) with the dimension of $\mathfrak{X}$ over $\mathfrak{D}$ greater than two; $\mathfrak{G}$ is a family of linear transformations on $\mathfrak{X}$ to $\mathfrak{X}$ each of which possesses an adjoint on $\mathfrak{X}^*$ [3, p. 16]; $\mathfrak{G}$ is a group under the "circle operation" $A \circ B = A + B - AB$; $\mathfrak{G}$ contains all one-dimensional involutions which possess adjoints. Denote by $\{\mathfrak{Y}, \mathfrak{E}, \mathfrak{Y}^* : \mathfrak{H}\}$ a second such system and assume that $\mathfrak{G}$ and $\mathfrak{H}$ are isomorphic as groups. In this situation, two principal results are obtained. The first states that one of the following pairs of linear space isomorphisms must hold: (i) $\mathfrak{X} \sim \mathfrak{Y}$ and $\mathfrak{X}^* \sim \mathfrak{Y}^*$ or (ii) $\mathfrak{X} \sim \mathfrak{Y}^*$ and $\mathfrak{X}^* \sim \mathfrak{Y}$. The second states that the group isomorphism is essentially generated by the corresponding linear space isomorphism.

G. Mackey [5, p. 251] has obtained the first result for the special case in which $\mathfrak{X}$, $\mathfrak{Y}$ are infinite dimensional normed linear spaces over the real numbers; $\mathfrak{X}^*$, $\mathfrak{Y}^*$ are the conjugate spaces of $\mathfrak{X}$, $\mathfrak{Y}$ respectively and $\mathfrak{G}$, $\mathfrak{H}$ are the (multiplicative) groups of all linear transformations on $\mathfrak{X}$, $\mathfrak{Y}$ respectively which are continuous and possess continuous inverses. Our proof is based on an extension of methods used by Mackey. The second result reduces essentially to a result of Dieudonné [1, Théorème 1] in the finite dimensional case. The finite dimensional case over a field has also been treated by Schreier and van der Waerden [6, p. 305].

In Section **1** are collected a few observations on dual spaces and involutions. Involutions are studied in detail in Sections **2**, **3**. The first result mentioned above is proved in Section **4** and the second is proved in Section **5**.

The groups studied here reduce essentially to full linear groups in the finite dimensional case. In a subsequent paper we will carry out a similar investigation for groups which reduce in finite dimensions to unitary, orthogonal or symplectic groups.

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results. In particular, a suggestion of his enabled us to complete the proof of Theorem 5.1.

1. Dual linear vector spaces. Involutions. Let $\mathfrak{X}$ denote a linear vector space over a division ring $\mathfrak{D}$. The space $\mathfrak{X}$ is called a *right* (*left*) $\mathfrak{D}$*-space* if the scalar multiplication is written on the right (left). A right $\mathfrak{D}$-space $\mathfrak{X}$ is said to be *isomorphic* to a right (left) $\mathfrak{E}$-space $\mathfrak{Y}$ provided there exists a one-to-one addition-preserving correspondence $x \leftrightarrow y$ between $\mathfrak{X}$ and $\mathfrak{Y}$ and an isomorphism (anti-isomorphism) $\alpha \leftrightarrow \beta$ between $\mathfrak{D}$ and $\mathfrak{E}$ such that $x\alpha \leftrightarrow y\beta$ ($x\alpha \leftrightarrow \beta y$). A similar definition holds with "right" and "left" interchanged.

Let $\mathfrak{X}$ and $\mathfrak{X}^*$ be right and left $\mathfrak{D}$-spaces respectively. Then $\mathfrak{X}$ and $\mathfrak{X}^*$ are said to be dual [3, p. 15] provided there exists a function (x^*, x) on $\mathfrak{X}^* \times \mathfrak{X}$ to $\mathfrak{D}$ with the following properties: (1) $(x^*, x) = 0$ for every $x^* \varepsilon \mathfrak{X}^*$ ($x \varepsilon \mathfrak{X}$) implies $x = 0$ ($x^* = 0$), (2) $(x^*, x\lambda + y\mu) = (x^*, x)\lambda + (x^*, y)\mu$ and $(\lambda x^* + \mu y^*, x) = \lambda(x^*, x) + \mu(y^*, x)$ for arbitrary $\lambda, \mu \varepsilon \mathfrak{D}$, $x, y \varepsilon \mathfrak{X}$ and $x^*, y^* \varepsilon \mathfrak{X}^*$.

Consider a subset $S \subset \mathfrak{X}$ ($S \subset \mathfrak{X}^*$) and define $S^{\perp}$ as the set of all $x^* \varepsilon \mathfrak{X}^*$ ($x \varepsilon \mathfrak{X}$) such that $(x^*, x) = 0$ for every $x \varepsilon S$ ($x^* \varepsilon S$). If the set S is finite, then the set $x^* + S^{\perp}$ ($x + S^{\perp}$) is called a neighborhood of $x^* \varepsilon \mathfrak{X}^*$ ($x \varepsilon \mathfrak{X}$). Under these neighborhood systems, $\mathfrak{X}^*$ and $\mathfrak{X}$ become T_1-spaces [3, p. 16]. If S consists of only one non-zero element, then $S^{\perp}$ is a maximal linear subspace of $\mathfrak{X}^*$ (or $\mathfrak{X}$). Let $x_1, \cdots, x_n \varepsilon \mathfrak{X}$ ($x^*_1, \cdots, x^*_n \varepsilon \mathfrak{X}^*$) be arbitrary linearly independent elements; then there exist elements $x^*_1, \cdots, x^*_n \varepsilon \mathfrak{X}^*$ ($x_1, \cdots, x_n \varepsilon \mathfrak{X}$) such that $(x^*_i, x_j) = \delta_{ij}$ [3, p. 16].

Let A be a linear transformation on $\mathfrak{X}$ to $\mathfrak{X}$. A linear transformation A^* on $\mathfrak{X}^*$ to $\mathfrak{X}^*$ is said to be *adjoint* to A provided $(x^*, xA) = (x^*A^*, x)$ for all $x \varepsilon \mathfrak{X}$ and $x^* \varepsilon \mathfrak{X}^*$. Whenever A^* exists it is unique, and a necessary and sufficient condition for the existence of A^* is that A be continuous in the T_1-topology defined above [3, p. 17]. The linear transformation A is said to be finite dimensional provided its range $\mathfrak{X}A$ is finite dimensional. Every continuous finite dimensional linear transformation A has the form $xA = \sum_{i=1}^{n} a_i(a^*_i, x)$, where $a_i \varepsilon \mathfrak{X}$ and $a^*_i \varepsilon \mathfrak{X}^*$ are fixed and n is the dimension of $\mathfrak{X}A$. Conversely, every finite dimensional transformation of this form is continuous and $x^*A^* = \sum_{i=1}^{n} (x^*, a_i)a^*_i$ defines its adjoint [3, p. 17].

In addition to the usual operations of addition and multiplication for linear transformations on a linear space, we also consider the *circle operation* $A \circ B = A + B - AB$ [4, p. 153; 2, p. 455]. This operation is associative

and has the identically zero transformation as an identity. The family of all continuous linear transformations on $\mathfrak{X}$ which have continuous inverses relative to the circle operation constitute a group under this operation. Moreover, under the correspondence $A \leftrightarrow I - A$, this group is isomorphic to the multiplicative group of all continuous linear transformations which have continuous inverses relative to ordinary multiplication. The inverse of a linear transformation A relative to the circle operation will be denoted by A°.

A linear transformation T is called an *involution* (relative to the circle operation) provided $T \circ T = 0$ or, equivalently, $T^2 = T2$. Observe that the adjoint of an involution is also an involution. For what follows, it is necessary to require the characteristic of $\mathfrak{D}$ to be different from 2.

1.1 THEOREM. *For each involution T on $\mathfrak{X}$ there exists a unique decomposition, $\mathfrak{X} = \mathfrak{M} + \mathfrak{N}$, $\mathfrak{M} \cap \mathfrak{N} = (0)$, of $\mathfrak{X}$ such that $xT = x2$ for $x \varepsilon \mathfrak{M}$ and $xT = 0$ for $x \varepsilon \mathfrak{N}$. If T^* exists, then the analogous decomposition of $\mathfrak{X}^*$ associated with T^* is $\mathfrak{X}^* = \mathfrak{N}^\perp + \mathfrak{M}^\perp$.*

Define $E = T\frac{1}{2}$ and observe that $E^2 = E$; that is, E is a projection on $\mathfrak{X}$. It is evident that the desired decomposition of $\mathfrak{X}$ is given by $\mathfrak{M} = \mathfrak{X}E$ and $\mathfrak{N} = \mathfrak{X}(I - E)$. If T^* exists, then so also does E^*; in fact $E^* = \frac{1}{2}T^*$. Since $x^*E^* = 0$ if, and only if, $(x^*, xE) = 0$ for every $x \varepsilon \mathfrak{X}$, it follows that $x^*E^* = 0$ is equivalent to $x^* \varepsilon \mathfrak{M}^\perp$. Also, since $x^*E^* = x^*$ if, and only if, $(x^*, x - xE) = 0$ for every $x \varepsilon \mathfrak{X}$, it follows that $x^*E^* = x^*$ is equivalent to $x^* \varepsilon \mathfrak{N}^\perp$. This completes the proof.

The subspaces $\mathfrak{M}$, $\mathfrak{N}$ associated with T by Theorem 1.1 are called the *subspaces* of T. The involution is said to be *minimal* provided $\mathfrak{M}$ is one-dimensional and *maximal* provided $\mathfrak{N}$ is one-dimensional. In either case, it is said to be *extremal*. Observe that the operation $T \circ I2$ interchanges the subspaces of T. Thus, if T is minimal (maximal), then $T \circ I2$ is maximal (minimal). If T_1, T_2 are two involutions with the same subspaces, then either $T_2 = T_1$ or $T_2 = T_1 \circ I2$.

The following lemma is an easy consequence of the definition of involution and the representation of continuous finite dimensional transformations mentioned above.

1.2. LEMMA. *Let T be a continuous involution on $\mathfrak{X}$. Then T is minimal if, and only if, it is of the form $xT = t2(t^*, x)$ and is maximal if, and only if, it is of the form $xT = x2 - t2(t^*, x)$, where $t \varepsilon \mathfrak{X}$ and $t^* \varepsilon \mathfrak{X}^*$ are fixed such that $(t^*, t) = 1$.*

Throughout the remainder of this paper, $\mathfrak{X}$ and $\mathfrak{X}^*$ will be dual $\mathfrak{D}$-spaces

where the characteristic of $\mathfrak{D}$ is different from 2 and $\dim(\mathfrak{X}) \geq 3$. Also $\mathfrak{G}$ will be a family of continuous linear transformations on $\mathfrak{X}$ which is a group under the circle operation and which contains all continuous minimal involutions. The family $\mathfrak{G}^*$ of adjoints of elements of $\mathfrak{G}$ is also a group under the circle operation. Moreover, the groups $\mathfrak{G}$ and $\mathfrak{G}^*$ are anti-isomorphic under the natural correspondence $A \leftrightarrow A^*$. A system of the type described here will be denoted by $\{\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^* : \mathfrak{G}\}$.

2. Intersection properties of extremal involutions. Consider the system $\{\mathfrak{X}, \mathcal{E}, \mathfrak{X}^* : \mathfrak{G}\}$ and denote by $\mathfrak{J}$ the set of all involutions in $\mathfrak{G}$. The objective of this section is to obtain, in terms only of the group operation in $\mathfrak{G}$, a necessary and sufficient condition that two non-commutative extremal involutions in $\mathfrak{J}$ shall have a subspace in common.

Let $\mathfrak{S}$ be an arbitrary subset of $\mathfrak{J}$ and denote by $c(\mathfrak{S})$ the set of all $T \,\varepsilon\, \mathfrak{J}$ such that $T \circ S = S \circ T$ for every $S \,\varepsilon\, \mathfrak{S}$. Observe that $c(\mathfrak{S})$ can also be defined as the set of all $T \,\varepsilon\, \mathfrak{J}$ such that $TS = ST$ for every $S \,\varepsilon\, \mathfrak{S}$, since $TS = ST$ is equivalent to $T \circ S = S \circ T$.

2.1. LEMMA. *Let T be an involution with subspaces $\mathfrak{M}$, $\mathfrak{N}$ and let A be an arbitrary linear transformation on $\mathfrak{X}$. Then $T \circ A = A \circ T$ if, and only if, $\mathfrak{M}A \subseteq \mathfrak{M}$ and $\mathfrak{N}A \subseteq \mathfrak{N}$.*

The necessity of the condition is trivial. Therefore assume $\mathfrak{M}A \subseteq \mathfrak{M}$, $\mathfrak{N}A \subseteq \mathfrak{N}$ and, for arbitrary $x \,\varepsilon\, \mathfrak{X}$, write $x = m + n$ where $m \,\varepsilon\, \mathfrak{M}$ and $n \,\varepsilon\, \mathfrak{N}$. Then $xTA = mA2$. Also, since $mA \,\varepsilon\, \mathfrak{M}$ and $nA \,\varepsilon\, \mathfrak{N}$, $xAT = mAT + nAT = mA2$. Therefore $TA = AT$ and hence $T \circ A = A \circ T$.

2.2 LEMMA. *Let T_i be involutions with subspaces $\mathfrak{M}_i$, $\mathfrak{N}_i$ $(i = 1, 2)$. A sufficient condition for $T_1 \circ T_2 = T_2 \circ T_1$ is that either $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ and $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$ or $\mathfrak{M}_1 \subseteq \mathfrak{N}_2$ and $\mathfrak{M}_2 \subseteq \mathfrak{N}_1$. If T_1 is minimal, then the condition is also necessary.*

Assume first that $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ and $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$. Obviously $\mathfrak{M}_1 T_2 \subseteq \mathfrak{M}_1$. Moreover, let $n_1 \,\varepsilon\, \mathfrak{N}_1$ and write $n_1 = m_2 + n_2$, where $m_2 \,\varepsilon\, \mathfrak{M}_2$ and $n_2 \,\varepsilon\, \mathfrak{N}_2$. Since $n_2 \,\varepsilon\, \mathfrak{N}_1$, we have $m_2 T_1 = 0$. Therefore, $n_1 T_2 T_1 = m_2 2 T_1 = 0$, so that $n_1 T_2 \,\varepsilon\, \mathfrak{N}_1$. In other words $\mathfrak{N}_1 T_2 \subseteq \mathfrak{N}_1$. That $T_1 \circ T_2 = T_2 \circ T_1$ now follows from Lemma 2.1. Sufficiency of the second condition is proved similarly. Now let $T_1 \circ T_2 = T_2 \circ T_1$ and assume T_1 minimal. By Lemma 2.1, $\mathfrak{M}_1 T_2 \subseteq \mathfrak{M}_1$. Therefore, since $\mathfrak{M}_1$ is one-dimensional, if u is a non-zero element of $\mathfrak{M}_1$, then $uT_2 = u\lambda$ where $\lambda \,\varepsilon\, \mathfrak{D}$. Since $T_2{}^2 = T_2 2$, this implies either $\lambda = 2$ and $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ or $\lambda = 0$ and $\mathfrak{M}_1 \subseteq \mathfrak{N}_2$. Suppose $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ and

let n_2 be an arbitrary element of $\mathfrak{N}_2$. Write $n_2 = m_1 + n_1$, where $m_1 \varepsilon \mathfrak{M}_1$ and $n_1 \varepsilon \mathfrak{N}_1$. Then $0 = n_2 T_2 T_1 = n_2 T_1 T_2 = m_1 2 T_2 = m_1 4$. Therefore $m_1 = 0$ and hence $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$. A similar proof holds in the case $\mathfrak{M}_1 \subseteq \mathfrak{N}_2$.

2.3 COROLLARY. *If T_1, T_2 are both minimal and commute, then either $T_1 = T_2$ or $T_1 T_2 = 0$.*

2.4 LEMMA. *Let T_i be non-commutative minimal involutions in $\mathfrak{J}$ with subspaces $\mathfrak{M}_i, \mathfrak{N}_i$ $(i = 1, 2)$. A necessary and sufficient condition for T to belong to $c(T_1, T_2)$ is that one subspace of T contain $\mathfrak{M}_1 + \mathfrak{M}_2$ and the other be contained in $\mathfrak{N}_1 \cap \mathfrak{N}_2$.*

The sufficiency is immediate from the first part of Lemma 2.2. On the other hand, if $T \varepsilon c(T_1, T_2)$ and has subspaces $\mathfrak{M}, \mathfrak{N}$, then by the second part of Lemma 2.2, either $\mathfrak{M}_i \subseteq \mathfrak{M}$ and $\mathfrak{N} \subseteq \mathfrak{N}_i$ or $\mathfrak{M}_i \subseteq \mathfrak{N}$ and $\mathfrak{M} \subseteq \mathfrak{N}_i$ for $i = 1, 2$. We have only to show that the same case occurs for both $i = 1, 2$. Therefore suppose $\mathfrak{M}_1 \subseteq \mathfrak{M}$, $\mathfrak{N} \subseteq \mathfrak{N}_1$, $\mathfrak{M}_2 \subseteq \mathfrak{N}$ and $\mathfrak{M} \subseteq \mathfrak{N}_2$. But this implies $\mathfrak{M}_1 \subseteq \mathfrak{N}_2$ and $\mathfrak{M}_2 \subseteq \mathfrak{N}_1$ so that T_1, T_2 commute, contrary to hypothesis.

2.5 LEMMA. *Let T_1, T_2 be non-commutative extremal involutions in $\mathfrak{J}$. A sufficient condition for $T \varepsilon c(c(T_1, T_2))$ is that one subspace of T be contained in the union of the one-dimensional subspaces of T_1, T_2 and the other contain the intersection of their non-one-dimensional subspaces. The condition is also necessary if* $\dim(\mathfrak{X}) > 3$ *or if* $\dim(\mathfrak{X}) = 3$ *and T_1, T_2 have a common subspace.*

Observe that there is no loss in assuming the T_i to be minimal with subspaces $\mathfrak{M}_i$, $\mathfrak{N}_i$ $(i = 1, 2)$. Let $T \varepsilon \mathfrak{J}$ satisfy the given condition and consider an arbitrary $T' \varepsilon c(T_1, T_2)$. By Lemma 2.4, one subspace of T' contains $\mathfrak{M}_1 + \mathfrak{M}_2$ and the other is contained in $\mathfrak{N}_1 \cap \mathfrak{N}_2$. That T commutes with T' and is therefore in $c(c(T_1, T_2))$ follows by Lemma 2.1. This proves the sufficiency.

Now assume $\dim(\mathfrak{X}) > 3$ and observe that, sinve $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are maximal, $\dim(\mathfrak{N}_1 \cap \mathfrak{N}_2) \geq 2$. Moreover, since $\dim(\mathfrak{M}_1 + \mathfrak{M}_2) \leq 2$ and $\mathfrak{M}_1 + \mathfrak{M}_2$ cannot be contained in $\mathfrak{N}_1 \cap \mathfrak{N}_2$, the set $\mathfrak{P}$ of elements of $\mathfrak{N}_1 \cap \mathfrak{N}_2$ which are not in $\mathfrak{M}_1 + \mathfrak{M}_2$ is non-vacuous. Also, if $z \varepsilon \mathfrak{P}$, then $z \neq 0$ and $z + \mathfrak{N}_1 \cap \mathfrak{N}_2 \cap (\mathfrak{M}_1 + \mathfrak{M}_2) \subseteq \mathfrak{P}$. It follows immediately that the linear subspace generated by $\mathfrak{P}$ must equal $\mathfrak{N}_1 \cap \mathfrak{N}_2$. Now take $T \varepsilon c(c(T_1, T_2))$ and let $\mathfrak{M}, \mathfrak{N}$ be its subspaces. Also let y, z be arbitrary elements of $\mathfrak{X}$ such that $y \not\varepsilon \mathfrak{M}_1 + \mathfrak{M}_2$ and $z \varepsilon \mathfrak{P}$. Since $\mathfrak{M}_1$, $\mathfrak{M}_2$ are one-dimensional and $y, z \not\varepsilon \mathfrak{M}_1 + \mathfrak{M}_2$, there exists $z^* \varepsilon \mathfrak{X}^*$ such that $(z^*, \mathfrak{M}_1 + \mathfrak{M}_2) = (0)$, $(z^*, z) = 1$, $(z^*, y) \neq 0$. Define Z by $xZ = z2(z^*, x)$. Then, by Lemma

2.4, $Z \varepsilon c(T_1, T_2)$ so that Z and T commute. By Lemma 2.2, one subspace of T must contain z and (z^*, x) must vanish for x in the other subspace, which must therefore exclude y. Holding z fixed in $\mathfrak{P}$ and allowing y to vary over elements of $\mathfrak{X}$ not in $\mathfrak{M}_1 + \mathfrak{M}_2$, we conclude that one subspace of T is contained in $\mathfrak{M}_1 + \mathfrak{M}_2$. On the other hand, by varying z over $\mathfrak{P}$, we see that the other subspace of T must contain $\mathfrak{P}$ and hence $\mathfrak{N}_1 \cap \mathfrak{N}_2$, by linearity. This completes the proof of the necessity for the case $\dim(\mathfrak{X}) > 3$.

If $\dim(\mathfrak{X}) = 3$, then it can happen that the set $\mathfrak{P}$ is vacuous. In this case $c(T_1, T_2)$ contains only 0 and $I2$ so that $c(c(T_1, T_2)) = \mathfrak{I}$ and the necessity obviously fails. On the other hand, if $\mathfrak{P}$ is not vacuous, the above proof of the necessity goes through. In particular, if T_1 and T_2 have a common subspace, then $\mathfrak{P}$ cannot be vacuous. This completes the proof.

2.6 THEOREM. *Let T_1, T_2 be arbitrary non-commutative extremal involutions in $\mathfrak{I}$. Then a necessary and sufficient condition for T_1, T_2 to have a subspace in common is that $c(c(U_1, U_2)) = c(c(T_1, T_2))$ for every pair of non-commutative extremal involutions $U_1, U_2 \varepsilon c(c(T_1, T_2))$.*

There is no loss in assuming T_i to be minimal with subspaces $\mathfrak{M}_i$, $\mathfrak{N}_i$ and written in the form $xT_i = t_i 2(t^*_i, x)$, $(t^*_i, t) = 1$ $(i = 1, 2)$. Note that T_1, T_2 will have a common subspace if, and only if, the elements in one, and only one, of the pairs t_1, t_2 and t^*_1, t^*_2 are linearly dependent.

Assume first that T_1, T_2 do not have a subspace in common. Define $u_1 = t_1 + t_2 - t_2(t^*_2, t_1)$; then u_1, t_2 are linearly independent and $(t^*_2, u_1) = 1$. Now define U_1 by $xU_1 = u_1 2(t^*_2, x)$ and take $U_2 = T_2$. If $\mathfrak{M}'_i$, $\mathfrak{N}'_i$ are the subspaces of U_i, then $\mathfrak{M}'_i \subseteq \mathfrak{M}_1 + \mathfrak{M}_2$ and $\mathfrak{N}'_i = \mathfrak{N}_2 \supseteq \mathfrak{N}_1 \cap \mathfrak{N}_2$ $(i = 1, 2)$. Hence, by Lemma 2.5, $U_i \varepsilon c(c(T_1, T_2))$ $(i = 1, 2)$. However, since $\mathfrak{N}'_i \cap \mathfrak{N}'_2 = \mathfrak{N}_2$ and $\mathfrak{N}_1 \neq \mathfrak{N}_2$, Lemma 2.5 also gives $T_1 \not\varepsilon c(c(U_1, U_2))$. Since $T_1 \varepsilon c(c(T_1, T_2))$, this proves the sufficiency.

Now assume that T_1, T_2 have a common subspace, say $\mathfrak{N}_1 = \mathfrak{N}_2$. Then t_1, t_2 are linearly independent. Let U_1, U_2 be any pair of non-commutative minimal involutions in $c(c(T_1, T_2))$ and let $xU_i = u_i 2(u^*_i, x)$ $(i = 1, 2)$. If $\mathfrak{M}'_i$, $\mathfrak{N}'_i$ are the subspaces of U_i $(i = 1, 2)$, it follows by Lemma 2.5 that $\mathfrak{N}'_i \supseteq \mathfrak{N}_1 \cap \mathfrak{N}_2$. Therefore $\mathfrak{N}'_1 = \mathfrak{N}'_2 = \mathfrak{N}_1 = \mathfrak{N}_2$. Also $\mathfrak{M}'_1 + \mathfrak{M}'_2 \subseteq \mathfrak{M}_1 + \mathfrak{M}_2$. But U_1, U_2 do not commute; therefore $\dim(\mathfrak{M}'_1 + \mathfrak{M}'_2) = 2$ so that $\mathfrak{M}'_1 + \mathfrak{M}'_2 = \mathfrak{M}_1 + \mathfrak{M}_2$. Another application of Lemma 2.5 gives $c(c(U_1, U_2)) = c(c(T_1, T_2))$. On the other hand, if $\mathfrak{M}_1 = \mathfrak{M}_2$, then the above argument applied to the adjoints T^*_i, U^*_i gives $c(c(U^*_1, U^*_2)) = c(c(T^*_1, T^*_2))$ and this evidently implies $c(c(U_1, U_2)) = c(c(T_1, T_2))$, completing the proof.

3. Group characterization of extremal involutions. The objective here is to characterize an extremal involution of $\mathfrak{I}$ in terms only of the group operation in $\mathfrak{G}$. Let $T_1, T_2 \in \mathfrak{I}$ and denote by $\nu(T_1, T_2)$ the number of distinct elements in $c(c(T_1, T_2))$. Define

$$\nu = \max \nu(T_1, T_2),\ T_1 \circ T_2 = T_2 \circ T_1; \qquad \nu_T = \max \nu(T, T'),\ T \circ T' = T' \circ T.$$

3.1 Lemma. *Let P be a continuous projection on $\mathfrak{X}$ with* $\dim(\mathfrak{X}P) > 1$ *and let Z be any linear transformation on $\mathfrak{X}$ which commutes with every minimal involution $U \in \mathfrak{I}$ such that $UP = PU = U$. Then there exists ζ in the center of $\mathfrak{D}$ such that $PZ = P\zeta$. If Z is an involution, then $\zeta = 2$ or* 0 *and furthermore the condition on the dimension of $\mathfrak{X}P$ can be dropped.*

Let u be an arbitrary non-zero element of $\mathfrak{X}P$ and choose $v^* \in \mathfrak{X}^*$ such that $(v^*, u) = 1$. Define $u^* = v^*P^*$; then also $(u^*, u) = 1$ and $xU = u2(u^*, x)$ defines a minimal involution $U \in \mathfrak{I}$ such that $UP = PU = U$. Since Z commutes with U, we have by Lemma 2.1 that $uZ = u\zeta(u)$, where $\zeta(u) \in \mathfrak{D}$. Now let v be any other element of $\mathfrak{X}P$. Then, since Z is additive, $(u + v)\zeta(u + v) = u\zeta(u) + v\zeta(v)$ and hence $u(\zeta(u + v) - \zeta(u)) = v(\zeta(v) - \zeta(u + v))$. If u, v are linearly independent, then $\zeta(u + v) = \zeta(u)$ and $\zeta(u + v) = \zeta(v)$ so that $\zeta(u) = \zeta(v)$. If u, v are linearly dependent, choose v' linearly independent of u, v. Then $\zeta(v') = \zeta(u)$ and $\zeta(v') = \zeta(v)$. Therefore $\zeta(u) = \zeta(v)$ in this case as well. In other words $\zeta(u)$ is equal to a constant ζ. That ζ is in the center of $\mathfrak{D}$ follows from the linearity of Z. Now, if Z is an involution, then $Z^2 = Z2$, which implies $\zeta(u) = 2$ or 0, for every $u \in \mathfrak{X}P$. Therefore, if $\zeta(u) \neq \zeta(v)$, then either $\zeta(u + v) = \zeta(u)$ or $\zeta(u + v) = \zeta(v)$. Since $u(\zeta(u + v) - \zeta(u)) = v(\zeta(v) - \zeta(u + v))$, the first implies $v = 0$ and the second implies $u = 0$. Hence, if u and v are not zero, then $\zeta(u) = \zeta(v)$. This completes the proof.

3.2 Lemma. *If* $\dim(\mathfrak{X}) > 3$, *then ν is equal to either* 16 *or* 8 *according as $I2 \in \mathfrak{G}$ or $I2 \notin \mathfrak{G}$.*

Let T_1, T_2 be any two commutative involutions in $\mathfrak{I}$ and set $E_i = T_i\frac{1}{2}$ $(i = 1, 2)$. Then E_i, $I - E_i$ are continuous projections on $\mathfrak{X}$. Since E_1, E_2 commute, the following are also continuous projections:

$$P_1 = E_1E_2,\ P_2 = E_1(I - E_2),\ P_3 = (I - E_1)E_2,\ P_4 = (I - E_1)(I - E_2).$$

Also $P_iP_j = 0$ for $i \neq j$. Observe that $c(T_1, T_2)$ consists of all $T \in \mathfrak{I}$ such that T commutes with each P_i.

Now let U_i denote any minimal involution in $\mathfrak{I}$ such that $P_iU_i = U_iP_i = U_i$. Clearly, for $j \neq i$, $U_iP_j = P_jU_i = 0$ so that $U_i \in c(T_1, T_2)$. By

Lemma 3.1, if $T \varepsilon c(c(T_1, T_2))$, then $P_i T = P_i \delta_i$, where $\delta_i = 2$ or 0 $(i = 1, 2, 3, 4)$. Since $I = \Sigma P_i$, we have $T = \Sigma P_i \delta_i$. Conversely, any T of this form is an involution and, provided it belongs to $\mathfrak{J}$, is in $c(c(T_1, T_2))$. It follows that $\nu \leq 16$, if $I2 \varepsilon \mathfrak{G}$, and a simple check shows that $\nu \leq 8$, if $I2 \not\varepsilon \mathfrak{G}$. It remains to prove equality.

Choose linearly independent elements $u_i \varepsilon \mathfrak{X}$ and $u^*_i \varepsilon \mathfrak{X}^*$ such that $(u^*_i, u_j) = \delta_{ij}$ $(i, j = 1, 2, 3)$. Define U_i by $xU_i = u_i 2(u^*_i, x)$; then $U_i \varepsilon \mathfrak{J}$ $(i = 1, 2, 3)$. Also define $T_1 = U_1 \circ U_2$ and $T_2 = U_1 \circ U_3$. Then T_1, T_2 commute and the associated projections P_i are given as follows: $P_1 = U_1 \frac{1}{2}$, $P_2 = U_2 \frac{1}{2}$, $P_3 = U_3 \frac{1}{2}$, $P_4 = I - (P_1 + P_2 + P_3)$. Note that $P_1 + P_2 + P_3 = (U_1 \circ U_2 \circ U_3) \frac{1}{2}$. Since $\dim(\mathfrak{X}) > 3$, none of the projections P_i is zero. It follows now that $\nu(T_1, T_2) = 16$, if $I2 \varepsilon \mathfrak{G}$, and $\nu(T_1, T_2) = 8$, if $I2 \not\varepsilon \mathfrak{G}$.

3.3 Theorem. *If* $\dim(\mathfrak{X}) \leq 3$, *then every involution on* $\mathfrak{X}$ *is extremal and* $\nu_T = \nu$ *for all* T. *If* $\dim(\mathfrak{X}) > 3$, *then a necessary and sufficient condition for* $T \varepsilon \mathfrak{J}$ *to be extremal is that* $\nu_T = \frac{1}{2}\nu$.

The first statement is obvious. by (2.3) To prove the second, consider an arbitrary $T_1 \varepsilon \mathfrak{J}$ with subspaces $\mathfrak{M}$, $\mathfrak{N}$. Let T_2 be any element of $\mathfrak{J}$ which commutes with T_1 and define P_i $(i = 1, 2, 3, 4)$ as in the proof of Lemma 3.2. If T_1 is minimal, then either $P_1 = 0$ or $P_2 = 0$ and, if T_1 is maximal, then $I2 \varepsilon \mathfrak{G}$ and either $P_3 = 0$ or $P_4 = 0$. In either case, it follows that $\nu(T_1, T_2) \leq \frac{1}{2}\nu$ and hence that $\nu_{T_1} \leq \frac{1}{2}\nu$. Now choose $u_1 \varepsilon \mathfrak{M}$, $u_2 \varepsilon \mathfrak{N}$ and $v^*_i \varepsilon \mathfrak{X}^*$ such that $(v^*_i, u_j) = \delta_{ij}$ $(i, j = 1, 2)$. Define $u^*_1 = v^*_1 E^*_1$ and $u^*_2 = v^*_2 (I - E_1)^*$, where $E_1 = T_1 \frac{1}{2}$. Then $(u^*_i, u_j) = \delta_{ij}$ $(i, j = 1, 2)$. If U_i is defined by $xU_i = u_i 2(u^*_i, x)$ $(i = 1, 2)$ and $T_2 = U_1 \circ U_2$, then $T_2 \varepsilon \mathfrak{J}$ and T_1, T_2 commute. With T_2 chosen in this way, since $\dim(\mathfrak{X}) > 3$, at most one of the associated projections P_i can be zero and this happens if, and only if, T_1 is extremal. It follows that $\nu(T_1, T_2) = \nu$ if T_1 is not extremal, and $\nu(T_1, T_2) = \frac{1}{2}\nu$ if T_1 is extremal. This completes the proof.

4. Isomorphic groups of transformations. In this section we consider two systems, $\{\mathfrak{X}, \mathcal{D}, \mathfrak{X}^* : \mathfrak{G}\}$ and $\{\mathfrak{Y}, \mathcal{E}, \mathfrak{Y}^* : \mathfrak{H}\}$, of the type discussed above, with $\mathfrak{J}$ and $\mathcal{J}$ as the sets of involutions in $\mathfrak{G}$ and $\mathfrak{H}$ respectively. We also assume throughout that the groups $\mathfrak{G}$ and $\mathfrak{H}$ are either isomorphic or anti-isomorphic. The isomorphism or anti-isomorphism of $\mathfrak{G}$ onto $\mathfrak{H}$ will be denoted by $\mathfrak{g}: G \to \mathfrak{g}(G)$. It is evident that $\mathfrak{g}$ sets up a one-to-one correspondence between the sets $\mathfrak{J}$ and $\mathcal{J}$. Also, $I2 \varepsilon \mathfrak{G}$ if, and only if, $I2 \varepsilon \mathfrak{H}$ and furthermore, if $I2 \varepsilon \mathfrak{G}$, then $\mathfrak{g}(I2) = I2$. By Theorem 3.3, an extremal involution must correspond under $\mathfrak{g}$ to an extremal involution. Moreover,

if T_i $(i = 1, \cdots, k)$ are arbitrary extremal involutions in $\mathfrak{I}$, then repeated application of Theorem 2. 6, plus the observation that two extremal involutions which commute and have a common subspace must have the same subspaces, shows that the T_i have a subspace in common if, and only if, the $\mathfrak{g}(T_i)$ do.

Let (T_1, T_2) and (T_3, T_4) be two pairs of non-commutative extremal involutions. Then the pairs are said to be *similar* provided each has a subspace in common and the common subspaces are either both one-dimensional or both non-one-dimensional.

4. 1 LEMMA. *The pairs (T_1, T_2) and (T_3, T_4) are similar if, and only if, $(\mathfrak{g}(T_1), \mathfrak{g}(T_2))$ and $(\mathfrak{g}(T_3), \mathfrak{g}(T_4))$ are similar.*

Let (T_1, T_2), (T_3, T_4) be similar and assume first that the common subspaces are one-dimensional. If these one-dimensional subspaces coincide, then the T_i $(i = 1, 2, 3, 4)$ have a common subspace. Therefore the $\mathfrak{g}(T_i)$ also have a common subspace and this implies the desired result. If the one-dimensional subspaces are different, let one be generated by u_1 and the other by u_2. Then u_1, u_2 are linearly independent and there exists $u^* \varepsilon \mathfrak{X}^*$ such that $(u^*, u_i) = 1$ $(i = 1, 2)$. Define U_i by $xU_i = u_i 2(u^*, x)$; then $U_i \varepsilon \mathfrak{I}$ $(i = 1, 2)$ and the involutions in each of the triples T_1, T_2, U_1 and T_3, T_4, U_2 have a common subspace. Moreover U_1, U_2 have a common subspace but (U_1, U_2) is not similar to either (T_1, T_2) or (T_3, T_4). Now suppose $(\mathfrak{g}(T_1), \mathfrak{g}(T_2))$ and $(\mathfrak{g}(T_3), \mathfrak{g}(T_4))$ not similar. Then one of these pairs, say the first, must be similar to the pair $(\mathfrak{g}(U_1), \mathfrak{g}(U_2))$. Since $\mathfrak{g}(T_1)$, $\mathfrak{g}(T_2)$, $\mathfrak{g}(U_1)$ have a subspace in common, it follows that $\mathfrak{g}(T_1)$, $\mathfrak{g}(T_2)$, $\mathfrak{g}(U_1)$, $\mathfrak{g}(U_2)$ have a common subspace. This implies in turn that T_1, T_2, U_1, U_2 have a common subspace and contradicts the fact that (T_1, T_2) and (U_1, U_2) are not similar. Therefore $(\mathfrak{g}(T_1), \mathfrak{g}(T_2))$, $(\mathfrak{g}(T_3), \mathfrak{g}(T_4))$ are similar. Finally if the common subspaces in (T_1, T_2) and (T_3, T_4) are not one-dimensional, then the above argument applied to (T^*_1, T^*_2) and (T^*_3, T^*_4) leads again to the desired result and completes proof of the necessity. The sufficiency clearly follows by symmetry.

The one-to-one correspondence between the sets of involutions $\mathfrak{I}$ and $\mathfrak{J}$ will now be used to construct a one-to-one correspondence between the one-dimensional subspaces of $\mathfrak{X}$ and those of either $\mathfrak{Y}$ or $\mathfrak{Y}^*$ such that linear dependence is preserved. The one-dimensional subspace of $\mathfrak{X}$ determined by a non-zero element $x \varepsilon \mathfrak{X}$ will be denoted by $x\mathfrak{D}$. The set of all one-dimensional subspaces of $\mathfrak{X}$ will be denoted by $\mathfrak{X}_{\mathfrak{D}}$. Similar notations will be used for the other spaces under consideration. If $x\mathfrak{D} \varepsilon \mathfrak{X}_{\mathfrak{D}}$ and $\mathfrak{D}x^* \varepsilon \mathfrak{X}^*_{\mathfrak{D}}$, then $x\mathfrak{D}$ and $\mathfrak{D}x^*$ are said to be *orthogonal* provided $(x^*, x) = 0$. If T is an

extremal involution in $\mathfrak{J}$, then there exist non-orthogonal $u\mathfrak{D} \varepsilon \mathfrak{X}_{\mathfrak{D}}$ and $\mathfrak{D}u^* \varepsilon \mathfrak{X}^*_{\mathfrak{D}}$ such that $u\mathfrak{D}$ and $(\mathfrak{D}u^*)^{\perp}$ are the subspaces of T. Conversely, every non-orthogonal pair $u\mathfrak{D}$, $\mathfrak{D}u^*$ is associated in this way with an extremal involution. If $I2 \notin \mathfrak{G}$, then this association is one-to-one and T is minimal. If $I2 \varepsilon \mathfrak{G}$, then both T and $I2 - T$ are associated with $u\mathfrak{D}$ and $\mathfrak{D}u^*$.

4.2 THEOREM. *There exist two one-to-one mappings F and F^* such that F maps $\mathfrak{X}_{\mathfrak{D}}$ onto either $\mathfrak{Y}_{\mathcal{E}}$ or $\mathfrak{Y}^*_{\mathcal{E}}$ while F^* maps $\mathfrak{X}^*_{\mathfrak{D}}$ onto either $\mathfrak{Y}^*_{\mathcal{E}}$ or $\mathfrak{Y}_{\mathcal{E}}$. Also $(x_3\mathfrak{D})F \subseteq (x_1\mathfrak{D})F + (x_2\mathfrak{D})F$ or $(\mathfrak{D}^*x_3)F^* \subseteq (\mathfrak{D}x^*_1)F^* + (\mathfrak{D}x^*_2)F^*$ if, and only if, $x_3\mathfrak{D} \subseteq x_1\mathfrak{D} + x_2\mathfrak{D}$ or $\mathfrak{D}x^*_3 \subseteq \mathfrak{D}x^*_1 + \mathfrak{D}x^*_2$ respectively.*

Since $\mathfrak{g}(T)$ is extremal if, and only if, T is extremal, it is evident that $\mathfrak{g}$ establishes a one-to-one correspondence $\{\mathfrak{D}x^*, x\mathfrak{D}\} \leftrightarrow \{\mathcal{E}y^*, y\mathcal{E}\}$ between non-orthogonal pairs from $\mathfrak{X}^*_{\mathfrak{D}} \times \mathfrak{X}_{\mathfrak{D}}$ and $\mathfrak{Y}^*_{\mathcal{E}} \times \mathfrak{Y}_{\mathcal{E}}$. Now let $\{\mathfrak{D}x^*, x\mathfrak{D}\}$ be an arbitrary non-orthogonal pair from $\mathfrak{X}^*_{\mathfrak{D}} \times \mathfrak{X}_{\mathfrak{D}}$ and let $\mathfrak{D}x^*_1$ be any element of $\mathfrak{X}^*_{\mathfrak{D}}$ distinct from $\mathfrak{D}x^*$ and also non-orthogonal to $x\mathfrak{D}$. Then $\{\mathfrak{D}x^*, x\mathfrak{D}\} \leftrightarrow \{\mathcal{E}y^*, y\mathcal{E}\}$ and $\{\mathfrak{D}x^*_1, x\mathfrak{D}\} \leftrightarrow \{\mathcal{E}y^*_1, y_1\mathcal{E}\}$. By Theorem 2.6, either $y_1\mathcal{E} = y\mathcal{E}$ or $\mathcal{E}y^*_1 = \mathcal{E}y^*$ and, by Lemma 4.1, whichever case holds is independent of the choice of $x\mathfrak{D}$, $\mathfrak{D}x^*$ and $\mathfrak{D}x^*_1$. Therefore we can define, without ambiguity, $(x\mathfrak{D})F = y\mathcal{E}$ if $y_1\mathcal{E} = y\mathcal{E}$, and $(x\mathfrak{D})F = \mathcal{E}y^*$ if $\mathcal{E}y^*_1 = \mathcal{E}y^*$. In the first case F maps $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$, and in the second it maps $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}^*_{\mathcal{E}}$. The mapping F^* is defined in an exactly similar way. It is not difficult to verify that F and F^* are one-to-one. Observe that $\{\mathfrak{D}x^*, x\mathfrak{D}\} \leftrightarrow \{(\mathfrak{D}x^*)F^*, (x\mathfrak{D})F\}$ or $\{\mathfrak{D}x^*, x\mathfrak{D}\} \leftrightarrow \{(x\mathfrak{D})F, (\mathfrak{F}x^*)F^*\}$ according as F maps $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$ or $\mathfrak{Y}^*_{\mathcal{E}}$. It remains to prove the last statement of the theorem. It will be sufficient to consider only F and prove that $x_3\mathfrak{D} \subseteq x_1\mathfrak{D} + x_2\mathfrak{D}$ implies $(x_3\mathfrak{D})F \subseteq (x_1\mathfrak{D})F + (x_2\mathfrak{D})F$. If $x_3\mathfrak{D}$ is equal to either $x_1\mathfrak{D}$ or $x_2\mathfrak{D}$, the desired result is immediate; therefore assume $x_3\mathfrak{D} \neq x_1\mathfrak{D}$ and $x_3\mathfrak{D} \neq x_2\mathfrak{D}$. Then also $x_1\mathfrak{D} \neq x_2\mathfrak{D}$. Choose $x_i \varepsilon x_i\mathfrak{D}$ so that $x_1 + x_2 \varepsilon x_3\mathfrak{D}$ and $x_3 = (x_1 + x_2)\frac{1}{2}$. Choose $x^* \varepsilon \mathfrak{X}^*$ such that $(x^*, x_1) = (x^*, x_2) = 1$; then also $(x^*, x_3) = 1$. Define involutions $X_i \varepsilon \mathfrak{J}$ as follows: $xX_i = x_i 2(x^*, x)$ $(i = 1, 2, 3)$. Observe that the subspaces of X_i are $x_i\mathfrak{D}$ and $(\mathfrak{D}x^*)^{\perp}$. Since $x_3\mathfrak{D} \subseteq x_1\mathfrak{D} + x_2\mathfrak{D}$, it follows, by Lemma 2, 5, that $X_3 \varepsilon c(c(X_1, X_2))$. Hence also $\mathfrak{g}(X_3) \varepsilon c(c(\mathfrak{g}(X_1), \mathfrak{g}(X_2)))$ and $\mathfrak{g}(X_3)^* \varepsilon c(c(\mathfrak{g}(X_1)^*, \mathfrak{g}(X_2)^*))$. Let $y_i\mathcal{E}$ and $(\mathcal{E}y^*_i)^{\perp}$ denote the subspaces of $\mathfrak{g}(X_i)$ $(i = 1, 2, 3)$. It follows from Lemma 2.5 that $y_3\mathcal{E} \subseteq y_1\mathcal{E} + y_2\mathcal{E}$. Similarly, by replacing $\mathfrak{g}(X_i)$ by $g(X_i)^*$, we obtain $\mathcal{E}y^*_3 \subseteq \mathcal{E}y^*_1$

$+ \mathcal{E}y^*_2$. Since $(x_i\mathfrak{D})F = y_i\mathcal{E}$ or $(x_i\mathfrak{D})F = \mathcal{E}y^*_i$, according as F maps $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$ or $\mathfrak{Y}^*_{\mathcal{E}}$, the proof is complete.

4.3 LEMMA. *Let $\mathfrak{M}$, $\mathfrak{N}$ be the subspaces of an involution $T \varepsilon \mathfrak{J}$. Then the linear subspaces determined by $(\mathfrak{M}_{\mathfrak{D}})F$ and $(\mathfrak{N}_{\mathfrak{D}})F$ in either $\mathfrak{Y}$ or $\mathfrak{Y}^*$ are the subspaces of the involution $\mathfrak{g}(T)$ or $\mathfrak{g}(T)^*$ according as F maps $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$ or $\mathfrak{Y}^*_{\mathcal{E}}$.*

It will be enough to consider the case in which F maps $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$. Let $\mathfrak{M}'$, $\mathfrak{N}'$, be the subspaces of $\mathfrak{g}(T)$ and consider the subspace $\mathfrak{M}$ of T. If $\mathfrak{M}$ is one-dimensional, then from the definition of F, we already know that $(\mathfrak{M}_{\mathfrak{D}})F$ is equal to either $\mathfrak{M}'_{\mathcal{E}}$ or $\mathfrak{N}'_{\mathcal{E}}$. If $\mathfrak{M}$ is not one-dimensional, let x_1, x_2 be any two linearly independent elements of $\mathfrak{M}$ and choose $v^* \varepsilon \mathfrak{X}^*$ such that $(v^*, x_i) = 1$ $(i = 1, 2)$. Let $u^* = \frac{1}{2}v^*T^*$, then still $(u^*, x_i) = 1$ and also $\mathfrak{N} \subseteq (\mathfrak{D}u^*)^{\perp}$. Define the involutions X_i by $xX_i = x_i 2(u^*, x)$ $(i = 1, 2)$. Then, by Lemma 2.2, $T \varepsilon c(X_1, X_2)$. Hence $\mathfrak{g}(T) \varepsilon c(\mathfrak{g}(X_1), \mathfrak{g}(X_2))$ and, by Lemma 2.4, either $\mathfrak{M}'$ or $\mathfrak{N}'$ must contain $(x_1\mathfrak{D})F + (x_2\mathfrak{D})F$. By holding $x_1 \varepsilon \mathfrak{M}$ fixed and varying x_2 over $\mathfrak{M}$, we conclude that either $\mathfrak{M}'_{\mathcal{E}}$ or $\mathfrak{N}'_{\mathcal{E}}$, say the first, must contain $(\mathfrak{M}_{\mathfrak{D}})F$. It follows easily by symmetry that $\mathfrak{M}'_{\mathcal{E}} = (\mathfrak{M}_{\mathfrak{D}})F$. A similar argument gives $\mathfrak{N}'_{\mathcal{E}} = (\mathfrak{N}_{\mathfrak{D}})F$ and completes the proof.

The next lemma appears to be well-known in modern projective geometry. However, since we do not have a reference for it, we will include a brief outline of a proof.

4.4 LEMMA. *Let $\mathfrak{X}$ be any right $\mathfrak{D}$-space with* $\dim(\mathfrak{X}) \geqq 3$ *and let $\mathfrak{Y}$ be a right or left $\mathcal{E}$-space. Also let F be any one-to-one mapping of $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$ such that $(x_3\mathfrak{D})F \subseteq (x_1\mathfrak{D})F + (x_2\mathfrak{D})F$ if, and only if, $x_3\mathfrak{D} \subseteq x_1\mathfrak{D} + x_2\mathfrak{D}$. Then there exists an isomorphic mapping Φ of $\mathfrak{X}$ onto $\mathfrak{Y}$ such that $(x\mathfrak{D})\Phi = (x\mathfrak{D})F$.*

It will be sufficient to take $\mathfrak{Y}$ as a right $\mathcal{E}$-space. Let x_0 be a fixed non-zero element of $\mathfrak{X}$ and denote by y_0 an arbitrary but fixed non-zero element of $(x_0\mathfrak{D})F$. We define $0\Phi_0 = 0$ and $x_0\Phi_0 = y_0$. If x is linearly independent of x_0, then x, $x_0 - x$ are linearly independent so that $(x\mathfrak{D})F \neq ((x_0 - x)\mathfrak{D})F$. Moreover, since $x_0\mathfrak{D} \subseteq x\mathfrak{D} + (x_0 - x)\mathfrak{D}$, we have $(x_0\mathfrak{D})F \subseteq (x\mathfrak{D})F + ((x_0 - x)\mathfrak{D})F$. Therefore there exist uniquely determined elements $y \varepsilon (x\mathfrak{D})F$ and $y' \varepsilon ((x_0 - x)\mathfrak{D})F$ such that $y_0 = y + y'$. Set $x\Phi_0 = y$. This defines Φ_0 for 0, x_0 and all x linearly independent of x_0. Now consider a second fixed element x_1 linearly independent of x_0 and define Φ_1 exactly as Φ_0 was defined except with x_0 replaced by x_1 and $x_1\Phi_1 = x_1\Phi_0$.

It turns out that $x\Phi_1 = x\Phi_0$ whenever both are defined. If $x\Phi = x\Phi_0$ or $x\Phi_1$, whichever is defined, then Φ is single-valued and can be shown to be additive. Now, for $\lambda \varepsilon \mathfrak{D}$, $(x_0\lambda)\Phi \varepsilon (x_0\mathfrak{D})F$. Therefore, there exists $\lambda^\phi \varepsilon \mathcal{E}$ such that $(x_0\lambda)\Phi = x_0\Phi\lambda^\phi$. Moreover it can be proved that $(x\lambda)\Phi = x\Phi\lambda^\phi$, for every x, and that $\lambda \rightarrow \lambda^\phi$ is an isomorphism of $\mathfrak{D}$ onto $\mathcal{E}$. Therefore Φ maps $\mathfrak{X}$ isomorphically onto $\mathfrak{Y}$ and it is obvious that $(x\mathfrak{D})\Phi = (x\mathfrak{D})F$.

4. 5 THEOREM. *There exist one-to-one mappings Φ, Ψ such that one of the following two cases is true:*

(i) *Φ and Ψ are continuous isomorphisms of $\mathfrak{X}$ onto $\mathfrak{Y}$ and $\mathfrak{X}^*$ onto $\mathfrak{Y}^*$ respectively, involving the same isomorphism $\lambda \rightarrow \lambda^\phi$ of $\mathfrak{D}$ onto $\mathcal{E}$ and such that $(x^*, y\Phi^{-1})^\phi \equiv (x^*\Psi, y)$.*

(ii) *Φ and Ψ are continuous isomorphisms of $\mathfrak{X}$ onto $\mathfrak{Y}^*$ and $\mathfrak{X}^*$ onto $\mathfrak{Y}$ respectively, involving the same anti-isomorphism $\lambda \rightarrow \lambda^\phi$ of $\mathfrak{D}$ onto $\mathcal{E}$ and such that $(y\Psi^{-1}, x)^\phi \equiv (x\Phi, y)$.*

In both cases $(x\mathfrak{D})\Phi = (x\mathfrak{D})F$ and $(\mathfrak{D}x^*)\Psi = (\mathfrak{D}x^*)F^*$, where F, F^* are the mappings given in Theorem 4. 2.

We will discuss only case (i) which corresponds to the case in which the mapping F of Theorem 4. 2 takes $\mathfrak{X}_{\mathfrak{D}}$ onto $\mathfrak{Y}_{\mathcal{E}}$. Denote by Φ an isomorphism of $\mathfrak{X}$ onto $\mathfrak{Y}$ given by Lemma 4. 4. By Lemma 4. 3, $(\mathfrak{D}x^*)F^* = \mathcal{E}y^*$ if, and only if, $(\mathfrak{D}x^*)\Phi = \mathcal{E}y^*$. It follows from this that, for fixed $x^* \varepsilon \mathfrak{X}^*$, $(x^*, y\Phi^{-1})^\phi$ defines an element of $(\mathfrak{D}x^*)F^*$ which we denote by $x^*\Psi$. It is not difficult to show that Ψ has the desired properties and also that both Φ, Ψ are continuous.

5. Representation of the group isomorphism. We turn in this section to the problem of representing $G \rightarrow \mathfrak{g}(G)$ in terms of the isomorphism of the underlying vector spaces given by Theorem 4. 5. It will be assumed throughout that $\mathfrak{g}$ is an isomorphism.

Consider the set of elements of the center of the division ring $\mathcal{E}$ minus the identity element 1. This set is a group under the circle operation and, as such, will be denoted by $\mathcal{E}_0$.

The following theorem provides the desired representation of the group isomorphism $\mathfrak{g}$. For the finite dimensional case, it reduces essentially to a theorem of Dieudonné [1, Théorème 1].

5. 1 THEOREM. *There exists a homomorphism $G \rightarrow \gamma(G)$ of $\mathfrak{G}$ into $\mathcal{E}_0$ such that one of the following is true for all $G \varepsilon \mathfrak{G}$:*

(i) $\mathfrak{g}(G) = (\Phi^{-1}G\Phi) \circ I\gamma(G)$; (ii) $\mathfrak{g}(G) = (\Phi^{-1}G^{\circ}\Phi)^{*} \circ I\gamma(G)$,

where Φ is the isomorphism given in the corresponding case of Theorem 4.5.

Consider first case (i) in which Φ is an isomorphism of $\mathfrak{X}$ onto $\mathfrak{Y}$. Let T be a minimal involution in $\mathfrak{G}$ with subspaces $\mathfrak{M}$ and $\mathfrak{N}$. Then, by Lemma 4.3, the involution $\mathfrak{g}(T)$ has $(\mathfrak{M})\Phi$ and $(\mathfrak{N})\Phi$ as its subspaces. On the other hand, $\Phi^{-1}T\Phi$ is also an involution with subspaces $(\mathfrak{M})\Phi$ and $(\mathfrak{N})\Phi$. It follows that $\mathfrak{g}(T) = (\Phi^{-1}T\mathfrak{N}) \circ I\rho$, where $\rho = 0$ or 2. Therefore $\mathfrak{g}(T)$ has the form (i) for minimal involutions.

We now define the mapping $G \rightarrow G^{\sigma} = \Phi\mathfrak{g}(G)\Phi^{-1}$, which is an isomorphism of the group $\mathfrak{G}$ onto another group of linear transformations in $\mathfrak{X}$. Observe that, if T is a minimal involution, then $T^{\sigma} = T \circ I\rho$, where $\rho = 0$ or 2.

Consider an arbitrary $G \varepsilon \mathfrak{G}$ and an arbitrary minimal involution $T \varepsilon \mathfrak{G}$. Then $G \circ T \circ G^{\circ}$ is also a minimal involution. In fact, $G \circ T \circ G^{\circ} = (I - G)T(I - G)^{-1}$, where $(I - G)^{-1}$ is the ordinary multiplicative inverse of $I - G$. Therefore it follows that $(G \circ T \circ G^{\circ})^{\sigma} = (G \circ T \circ G^{\circ}) \circ I\rho_1$, where $\rho_1 = 0$ or 2. But $(G \circ T \circ G^{\circ})^{\sigma} = G^{\sigma} \circ T^{\sigma} \circ (G^{\sigma})^{\circ}$ and $T^{\sigma} = T \circ I\rho_2$, where $\rho_2 = 0$ or 2. Hence $G^{\sigma} \circ T \circ (G^{\sigma})^{\circ} = (G \circ T \circ G^{\circ}) \circ I\rho$, where $\rho = \rho_1 \circ \rho_2 = 0$ or 2. This last equation can be written in the form $I\rho = (I - G^{\sigma})T(I - G^{\sigma})^{-1} + (I - G)T(I - G)^{-1}(\rho - 1)$. Each term on the right is one-dimensional and consequently the right hand side is at most two-dimensional. Since $\mathfrak{X}$ is at least three-dimensional, we conclude that $\rho = 0$. In other words,

$$(1) \qquad (I - G^{\sigma})T(I - G^{\sigma})^{-1} = (I - G)T(I - G)^{-1}.$$

Suppose now that there exists an $x \varepsilon \mathfrak{X}$ such that the vectors $u = x(I - G)$ and $v = x(I - G^{\sigma})$ are linearly independent. We can then choose T such that $uT = u2$ and $vT = 0$. An application of equation (1) to the vector x then yields $u2(I - G)^{-1} = 0$. This obviously implies $u = 0$, a contradiction. It follows that, for every $x \varepsilon \mathfrak{X}$, the vectors $x(I - G^{\circ})$ and $x(I - G)$ are linearly dependent. An argument similar to the above also shows that these vectors vanish simultaneously. Therefore, for every $x \varepsilon \mathfrak{X}$, there exists $\lambda_x \varepsilon \mathfrak{D}$ such that $x(I - G^{\sigma}) = x(I - G)\lambda_x$. Hence $x(I - G^{\sigma})(I - G)^{-1} = x\lambda_x$. By additivity, we obtain $x(\lambda_x - \lambda_{x+y}) + y(\lambda_y - \lambda_{x+y}) = 0$, for all $x, y \varepsilon \mathfrak{X}$. If x, y are linearly independent, this implies $\lambda_x = \lambda_{x+y} = \lambda_y$. If x, y are linearly dependent, choose $z \varepsilon \mathfrak{X}$ linearly independent of x, y. Then $\lambda_x = \lambda_z = \lambda_y$ so that $\lambda_x = \lambda_y$ in this case as well. Therefore λ_x is equal to a constant λ independent of x. In other words, $I - G^{\sigma} = (I - G)\lambda$, and

this can be written in the form $G^\sigma = G \circ I(1-\lambda)$. It is immediate from linearity that λ is in the center of $\mathfrak{D}$. Also, since $I(1-\lambda) = G^\circ \circ G^\sigma$ and $(G^\circ \circ G^\sigma)^\circ$ exists, it follows that $1-\lambda$ is in $\mathfrak{D}_0$. Returning to $\mathfrak{g}(G)$, we have $\mathfrak{g}(G) = (\Phi^{-1}G\Phi) \circ I\gamma(G)$, where $\gamma(G) = (1-\lambda)^\phi$ and $\lambda \rightarrow \lambda^\phi$ is the isomorphism of $\mathfrak{D}$ onto $\mathcal{E}$ associated wth Φ. It is obvious that $G \rightarrow \gamma(G)$ must be a homomorphism of $\mathfrak{G}$ into $\mathcal{E}_0$. This completes the proof for case (i).

In case (ii) we consider $G \rightarrow G^\delta = \Phi\mathfrak{g}(G)^*\Phi^{-1}$ instead of $G \rightarrow G^\sigma$. Observe that $G \rightarrow G^\delta$ is an anti-isomorphism of $\mathfrak{G}$. An argument similar to that given in case (i) yields, in place of equation (1),

$$(1)^* \qquad (I-G^\delta)^{-1}T(I-G^\delta) = (I-G)T(I-G)^{-1}.$$

Again as in case (i), we obtain that $I-G = (I-G^\delta)^{-1}\lambda$, where $\lambda \,\varepsilon\, \mathfrak{D}_0$. This can be written in the form $G^\delta = G^\circ \circ I(1-\lambda)$. Therefore

$$\mathfrak{g}(G) = (\Phi^{-1}G^\circ\Phi)^* \circ I\gamma(G),$$

where $\gamma(G) = (1-\lambda)^\phi$. This completes the proof.

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STEADY, ROTATIONAL, PLANE FLOW OF A GAS.* [1]

By Monroe H. Martin.

1. Introduction. If one excludes flows for which the stream lines are isobars, the network formed by the stream lines and isobars in the region of the (x, y)-plane covered by the flow can be used as a system of curvilinear coordinates. This amounts to taking the pressure p and stream function ψ for independent variables and to regard the other flow variables, i. e., the velocity components u, v, the density ρ, and the coordinates x, y, as unknown functions of them.

This choice of independent variables has the following advantage. Taking the equation of state of the gas to be $\rho = f(p, S)$, where S denotes the entropy, it is well known that S is constant along a stream line. Consequently, if we specify the variation of entropy from stream line to stream line by setting $S = S(\psi)$, the density ρ becomes a known function $\rho = \rho(p, \psi)$ of the independent variables p, ψ.

The problem of determining the remaining unknown functions x, y, u, v of p, ψ is reduced in Section **4** to the integration of a quasi-linear partial differential equation for a single unknown function $\theta = \arctan v/u = \theta(p, \psi)$, provided $\theta_\psi \not\equiv 0$, that is, provided the flow does not cross each isobar with a constant direction. These exceptional flows are studied in Section **5**.

The quasi-linear partial differential equation for θ involves the speed $q = q(p, \psi)$ of the flow, which becomes a known function of p, ψ, once $\rho(p, \psi)$ is given and two arbitrary functions $p_0 = p_0(\psi)$, $q_0 = q_0(\psi)$ are specified. It is, however possible to adopt an inverse approach, and assign θ to be a definite function of p, ψ. This leads to a *linear partial differential equation of the third order* for q. As an example of the inverse method, in Section **6** we study the simplest case, $\theta = \theta(\psi)$, in which the stream lines lie on straight lines.

2. The equations of motion. The dynamical equations and the equations of continuity in steady, plane flow of a fluid subject to no external forces constitute an underdetermined system

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[1] Sponsored by the Office of Naval Research. Presented April 2, 1949 at the Durham, North Carolina meeting of the American Mathematical Society. This investigation was carried out under project NOL 139 of the Naval Ordnance Laboratory, White Oak, Maryland.

$$\rho(u_x u + u_y v) + p_x = 0, \qquad \rho(v_x u + v_y v) + p_y = 0, \qquad (\rho u)_x + (\rho v)_y = 0, \tag{1}$$

of three partial differential equations for four unknown functions u, v, ρ, p of x, y. Following the usual notation, u, v designate the *components of the velocity*, ρ the *density*, and p the *pressure* of the fluid at the point (x, y).

Writing this system in the alternative form

$$(p + \rho u^2)_x + (\rho uv)_y = 0, \qquad (\rho uv)_x + (p + \rho v^2)_y = 0, \qquad (\rho u)_x + (\rho v)_y = 0,$$

it is easy to see that, given any solution

$$u = u(x, y), \qquad v = v(x, y), \qquad \rho = \rho(x, y) \neq 0, \qquad p = p(x, y), \tag{2}$$

of (1), there exist three functions[2] $\bar{\xi}$, $\bar{\eta}$, ψ of x, y defined by

$$d\bar{\xi} = -\rho uv dx + (p + \rho u^2) dy, \qquad d\bar{\eta} = -(p + \rho v^2) dx + \rho uv dy, \qquad d\psi = -\rho v dx + \rho u dy. \tag{3}$$

This differential system is replaced by the system

$$d\xi = -y dp + u d\psi, \qquad d\eta = x dp + v d\psi, \qquad d\psi = -\rho v dx + \rho u dy, \tag{4}$$

when we set $\xi = \bar{\xi} - py$, $\eta = \bar{\eta} + px$.

The form of the first two equations in (4) suggests that p, ψ be employed as independent variables, that is, that the stream lines and isobars be taken as a system of curvilinear coordinates in the region of the (x, y)-plane covered by the flow. We accordingly assume that

$$0 < |J^{-1}| < +\infty, \qquad J^{-1} = \partial(p, \psi)/\partial(x, y), \tag{5}$$

holds throughout the above region. This automatically rules out flows for which the pressure is constant everywhere or for which $p = p(\psi)$, i. e., flows for which the stream lines are isobars. Such flows form a very restricted class[3] and will be excluded from now on. When p, ψ are taken for independent variables, (4) leads to an underdetermined system of four partial differential equations

$$u_p = -y_\psi, \qquad v_p = x_\psi, \qquad u y_p - v x_p = 0; \qquad u y_\psi - v x_\psi = \rho^{-1}, \tag{6}$$

for five unknown functions x, y, u, v, ρ of p, ψ. The first two equations

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[2] ψ is, of course, the stream function. For the physical significance of $\bar{\xi}$, $\bar{\eta}$ see a Technical Memorandum, *A new approach to problems in two-dimensional flow.* NOLM 9869, Naval Ordnance Laboratory, by the author. To appear in Quarterly of Applied Mathematics.

[3] For a treatment of flows for which $p = p(\psi)$ see the memorandum cited in 2.

represent the integrability conditions for the first two equations in (4). The last two equations arise from the third equation of (4) when one sets $dx = x_p dp + x_\psi d\psi$, $dy = y_p dp + y_\psi d\psi$, and recalls that p, ψ are independent variables.

Starting with a solution (2) of the underdetermined system (1), one uses the last equation in (3) to calculate $\psi = \psi(x, y)$. The pair of equations $p = p(x, y)$, $\psi = \psi(x, y)$, determine x, y as functions of p, ψ, in view of (5). It follows from (2) that u, v, ρ are known functions of p, ψ. Thus, given any solution (2) of the underdetermined system (1), we are led to a solution

$$(7) \quad x = x(p, \psi),\ y = y(p, \psi),\ u = u(p, \psi),\ v = v(p, \psi),\ \rho = \rho(p, \psi) \neq 0,$$

of the underdetermined system (6).

Conversely, given a solution (7) of (6), a solution (2) of (1) can be constructed, provided

$$(8) \qquad 0 < | J | < +\infty, \qquad J = \partial(x, y)/\partial(p, \psi).$$

In view of (8), the first two equations in (7) can be solved for p, ψ. This yields p, ψ and consequently u, v, ρ as functions of x, y. One has, of course, $J\psi_x = -y_p$, $J\psi_y = x_p$, $Jp_x = y_\psi$, $Jp_y = -x_\psi$, and therefore

$$(9) \quad \begin{array}{lll} Ju_x = -u_\psi y_p + u_p y_\psi, & Jv_x = -v_\psi y_p + v_p y_\psi, & J\rho_x = -\rho_\psi y_p + \rho_p y_\psi, \\ Ju_y = u_\psi x_p - u_p x_\psi, & Jv_y = v_\psi x_p - v_p x_\psi, & J\rho_y = \rho_\psi x_p - \rho_p x_\psi. \end{array}$$

When the partial derivatives of u, v, ρ, p with respect to x, y obtained from these equations are substituted in the system (1), it will be found that the system is satisfied by virtue of (6) and the further relation $\rho_p = \rho^2(u_\psi y_p - u_p y_\psi + v_p x_\psi - v_\psi x_p)$. This relation is needed to verify that the continuity equation is satisfied and is obtained from the second column of (6). The first equation is differentiated with respect to ψ, the second with respect to p and the results are subtracted.

To examine the significance of (8), we find from (6) that

$$(10) \qquad u u_p + v v_p = -\rho^{-1},$$

and that $u_p x_p + v_p y_p = -J$, $v x_p - u y_p = 0$. Solving these equations simultaneously for x_p, y_p, and using (10), one finds

$$(11) \qquad x_p = \rho J u, \qquad y_p = \rho J v.$$

It follows that

$$(11') \qquad dp/ds = (\rho J q)^{-1},\ \text{if}\ q = (u^2 + v^2)^{\frac{1}{2}} > 0,$$

the arc length s being measured along a stream line from a fixed point so that it increases in the direction of flow on the stream line. Thus, provided

we assume that the density ρ and speed q remain finite and different from zero, the two conditions $0 < | dp/ds | < +\infty$, $0 < | J | < +\infty$, are equivalent.

The above results are summed up in the theorem below.

THEOREM 1. *The solution of the underdetermined systems* (1) *and* (6) *are equivalent problems, provided the stream lines are not isobars, the density,* ρ, *the speed* q, *and the pressure gradient* dp/ds *along a stream line remain finite and never vanish.*

The system (6) becomes determinate when the equation of state of the gas and the distribution of entropy S on the stream lines is specified by setting $\rho = f(p, S)$, $S = S(\psi)$.

For a polytropic gas, for example, one places

$$\rho = kp^{1/\gamma}, \qquad k = e^{-(S-S_0)/c_p}, \qquad S = S(\psi), \gamma > 1, \tag{12}$$

the function $S(\psi)$ being taken arbitrarily. After the equation of state of the gas and the distribution of entropy on the stream lines has been fixed, the density ρ becomes a known function

$$\rho = \rho(p, \psi), \tag{13}$$

which we term the *density function.* Once it has been assigned, (6) becomes a determinate system of four partial differential equations for four unknown functions x, y, u, v of p, ψ.

A solution

$$\begin{aligned} x &= x(p, \psi), \qquad & u &= u(p, \psi), \\ y &= y(p, \psi), \qquad & v &= v(p, \psi), \end{aligned} \tag{14}$$

of a determinate system (6) offers a parametric representation of the stream lines and their hodographs in the first and second columns respectively, the pressure p serving for parameter.

The functions in (14) are not necessarily single valued, as it is possible for a point in the (p, ψ)-plane to correspond to more than one point in the physical plane. Such a situation will always arise when an isobar cuts a fixed stream line in more than one point and occurs even in the simplest examples, e. g., the flow [4] defined by the conformal mapping $z = w^2$ in which the stream lines are confocal parabolas and the isobars are circles concentric at the common focus.

Moreover it is also possible for two different points in the (p, ψ)-plane

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[4] As the referee has pointed out, another example of great practical importance occurs in the neighborhood of the stagnation point in steady flow about a symmetrical blunt profile.

to map into the same point of the physical plane. This physical impossibility may be avoided by confining the point (p, ψ) to a properly chosen region in the (p, ψ)-plane.

3. The Bernoulli function. If (10) be integrated between the limits p_0 and p for a fixed ψ, we obtain Bernoulli's theorem

$$\tfrac{1}{2}q^2 = \tfrac{1}{2}q_0{}^2 - \int_{p_0}^{p} \rho^{-1}dp, \qquad p_0 = p_0(\psi),\ q_0 = q_0(\psi), \tag{15}$$

where the arbitrary functions $p_0(\psi)$, $q_0(\psi)$ are at our disposal. To understand their significance, let us imagine the region covered by the flow referred to its stream lines and isobars as a system of curvilinear coordinates, as in the figure below.

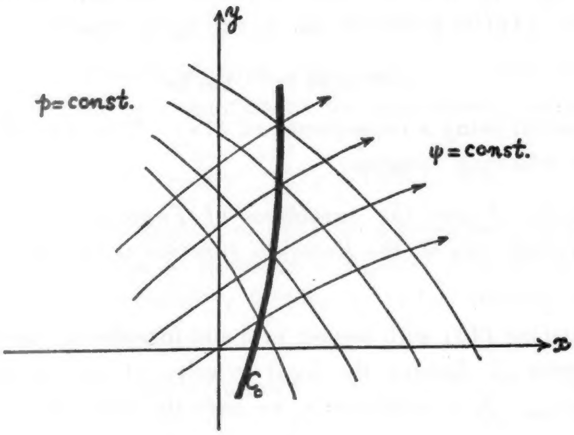

Figure 1.

The choice of $p_0(\psi)$ selects a curve $p = p_0(\psi)$ (the curve C_0) along which the speed of flow is determined by specifying $q_0(\psi)$. In particular, if $p_0 = \text{const.}$, C_0 is an isobar.

For a polytropic gas we find, from (12), that (15) yields

$$q^2 = q_0{}^2 - 2(p^{1-n} - p_0{}^{1-n})/\{k(1-n)\}, \qquad 0 < n = 1/\gamma < 1. \tag{16}$$

Taking $p_0 = 0$, this reduces to

$$q^2 = q_0{}^2 - 2p^{1-n}/\{k(1-n)\}, \tag{17}$$

and C_0 becomes the isobar of zero pressure along which the speed attains its maximum value q_0 (the ultimate speed). If in addition, $q_0 = \text{const.}$, the flow is iso-energetic.

Returning to the general case, we assume that the density function (13) and the arbitrary functions $p_0(\psi)$, $q_0(\psi)$ are specified. From (15) we see

$$q = q(p, \psi) \tag{18}$$

is a known function of p, ψ. This function is termed the *Bernoulli function* of the flow.

Conversely, if the Bernoulli function is given, the density function is uniquely determined by

$$\rho = -(qq_p)^{-1}, \tag{19}$$

provided, of course, that $qq_p < 0$.

An interesting connection between the Bernoulli function q and the vorticity $\omega = u_y - v_x$ follows[5] from (9). In substituting for u_y, v_x and then using (6), (11) to substitute for u_p, v_p, x_p, y_p, one has

$$\omega = \rho q q_\psi = - q_\psi / q_p, \tag{20}$$

the latter equation being a consequence of (19). This formula leads immediately to the following theorem:

THEOREM 2. *Under the hypotheses of Theorem* 1, *the flow will be irrotational if, and only if, the Bernoulli function is independent of* ψ, *that is* $q = q(p)$.

Differentiating (19) with respect to p and introducing the Mach number $M = q/G$, where G denotes the local velocity of sound, one finds that $M^2 - 1 = \rho^2 q^3 q_{pp}$. As a consequence, we have the theorem:

THEOREM 3. *Under the hypotheses of Theorem* 1, *the flow is subsonic, or supersonic, according as the Bernoulli function satisfies the inequality* $q_{pp} < 0$, *or the inequality* $q_{pp} > 0$.

As a corollary we note that if q is a linear function of p, the flow is always sonic.

4. The quasi-linear partial differential equation. The integration of a determinate system (6) may be reduced to the integration of a quasi-linear partial differential equation.

Let (14) be a solution of (6) and let $q = (u^2 + v^2)^{\frac{1}{2}} = q(p, \psi)$ be the associated Bernoulli function. If we introduce the angle of inclination θ of the stream lines by setting

[5] For a different proof see 2.

(21) $$u = q\cos\theta, \qquad v = q\sin\theta,$$

it follows from (11) that $\theta = \arctan(y_p/x_p)$ and, from (11′), that

(22) $$x_p = (dp/ds)^{-1}\cos\theta, \qquad y_p = (dp/ds)^{-1}\sin\theta.$$

Moreover, from (6), (11) and (21) we find

(23) $$\theta_\psi = -(q_{pp} - q\theta_p{}^2)\,dp/ds.$$

If we rule out temporarily those flows for which $\theta_\psi \equiv 0$, we may use (23) to eliminate dp/ds from (22) to obtain

(24) $$x_p = -\{(q_{pp} - q\theta_p{}^2)/\theta_\psi\}\cos\theta, \qquad y_p = -\{(q_{pp} - q\theta_p{}^2)/\theta_\psi\}\sin\theta.$$

On the other hand we find from (6) and (21) that

(25) $$x_\psi = (q\sin\theta)_p, \qquad y_\psi = -(q\cos\theta)_p.$$

When x and y are eliminated from (24), (25) by partial differentiation, it follows that θ must be a solution of the quasi-linear partial differential equation

(26) $$q(\{q_{pp} - q\theta_p{}^2\}/\theta_\psi)_\psi + (q^2\theta_p)_p = 0.$$

Conversely, if θ is a solution of (26) for a prescribed Bernoulli function $q = q(p,\psi)$, it is possible to construct a solution (14) of the determinate system (6) in which the density ρ is given by (19). Guided by (21), (24), (25) one tentatively writes down

(27) $$\begin{aligned} x &= \int\{-[(q_{pp} - q\theta_p{}^2)/\theta_\psi]\cos\theta\,dp + (q\sin\theta)_p d\psi\}, & u &= q\cos\theta, \\ y &= \int\{-[(q_{pp} - q\theta_p{}^2)/\theta_\psi]\sin\theta\,dp - (q\cos\theta)_p d\psi\}, & v &= q\sin\theta, \end{aligned}$$

and then verifies that this is indeed a solution of (6) by direct substitution.

We return to the case $\theta_\psi \equiv 0$ excluded above. For such flows $\theta = \theta(p)$, i.e. the velocity vectors are all parallel along an isobar. These flows will be examined in detail in Section **5**. At present we are only interested in seeing how (26), (27) must be modified to account for them.

From (23) it follows that (26) is replaced by

(28) $$q_{pp} - q\theta_p{}^2 = 0.$$

In place of (22) we write

(29) $$x_p = \lambda\cos\theta, \qquad y_p = \lambda\sin\theta, \qquad \lambda = \lambda(p,\psi),$$

and eliminating x, y from (25), (29) by partial differentiation, one finds

(30) $$\lambda = \int_{\psi_0}^{\psi}(q^2\theta_p)_p/q\,d\psi + B(p), \qquad \psi_0 = \text{const.},$$

where $B(p)$ denotes an arbitrary function. Consequently, in place of (27), we have

$$\text{(31)} \qquad \begin{aligned} x &= \int \{\lambda \cos\theta \, dp + (q \sin\theta)_p d\psi\}, & u &= q\cos\theta; \\ y &= \int \{\lambda \sin\theta \, dp - (q \cos\theta)_p d\psi\}, & v &= q\sin\theta. \end{aligned}$$

Conversely, if $\theta = \theta(p)$ satisfies (28) for a given Bernoulli function $q = q(p, \psi)$, and λ is defined by (30), the line integrals in (31) define x, y as functions of p, ψ. By direct substitution we verify that (31) is a solution (14) of (6) with the density function ρ given by (19).

We summarize these results in the next theorem.

THEOREM 4. *If the Bernoulli function $q(p, \psi)$ of the flow is given, the stream lines and their hodographs are presented parametrically in terms of the pressure p by* (27), *where θ is a solution of the quasi-linear equation* (26), *provided the direction of flow is not constant along the isobars, in which event* (27) *must be replaced by* (31), *and* (26) *by* (28).

When the partial differentiations are carried out in (26) this equation becomes

$$\text{(32)} \quad (q_{pp} - q\theta_p{}^2)\theta_{\psi\psi} + 2q\theta_p\theta_\psi\theta_{p\psi} - q\theta_\psi{}^2\theta_{pp} - (q_{pp\psi} - q_\psi\theta_p{}^2 + 2q_p\theta_p\theta_\psi)\theta_\psi = 0.$$

This brings out the quasi-linear character of (26).

5. The case $\theta = \theta(p)$. Equation (28) may be made to play a dual rôle. Depending on whether one elects to preassign the Bernoulli function q or the function θ, it becomes an equation for θ, or an equation for q.

Let us take the equation in the first rôle under the hypothesis that the flow is irrotational. According to Theorem 2 we prescribe

$$\text{(33)} \qquad q = q(p),$$

and, on solving (28) for θ, obtain

$$\text{(34)} \qquad \theta = \theta(p) = \int_{p_0}^{p} (q_{pp}/q)^{\frac{1}{2}} dp + \theta_0, \qquad \theta_0 = \text{const.}$$

For θ to be real we admit only values of p for which $q_{pp} \geqq 0$ and therefore, from Theorem 3, exclude subsonic flows.

Substituting from (33), (34) into (31), the second column yields the parametric equations of a one-parameter family of curves (with parameter θ_0) in the hodograph plane, generated by rotating any member h about the origin. A member h of this family is the hodograph of a flow in the physical plane, i. e., the flow in the physical plane has a one-dimensional hodograph.

As is the case for all irrotational flows, it follows from (33) that the isobars in the hodograph plane are circles concentric with the origin.

To construct the stream lines in the physical plane, we carry out the integration of the line integrals in the first column of (31) along the broken line P_0AP indicated below.

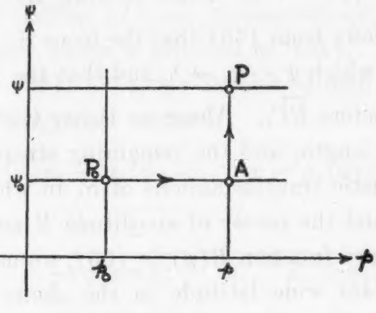

Figure 2.

One finds

(35) $$x = x_0(p) + (\psi - \psi_0)v', \qquad y = y_0(p) - (\psi - \psi_0)u', \quad ' = d/dp,$$

where

(36) $$x_0(p) = \int_{p_0}^{p} B\cos\theta\, dp, \qquad y_0(p) = \int_{p_0}^{p} B\sin\theta\, dp,$$

in which $\theta = \theta(p)$ is defined in (34), and B is the arbitrary function of p introduced in (30).

Setting $\psi = \psi_0$ in (35), one obtains an "initial stream line" S_0

(37) $$x = x_0(p), \qquad y = y_0(p),$$

containing the origin of the physical plane as a point at which $p = p_0$, $\theta = \theta_0$. Once S_0 has been drawn, the other stream lines may be constructed as explained below.

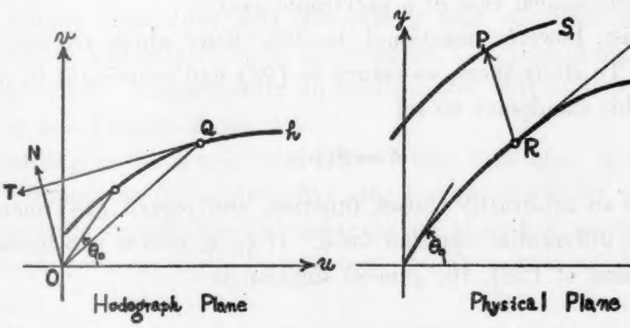

Figure 3.

Through a point R of S_0 draw the tangent RR' and from the origin O of the hodograph plane draw the radius vector $\overrightarrow{OQ}$ parallel to RR' terminating at Q on h. $\overrightarrow{OQ}$ represents the velocity vector at R. At Q draw the tangent vector $\overrightarrow{QT}$ to h with components (u', v') and from O draw the normal vector $\overrightarrow{ON}$ with components $(v', -u')$. From R draw $\overrightarrow{RP_1}$ parallel and equal in length to $\overrightarrow{ON}$. It follows from (35) that the locus S_1 of P_1 as R traverses S_0 is the stream line for which $\psi - \psi_0 = 1$, and that the isobars are the straight line containing the vectors $\overrightarrow{RP_1}$. Along an isobar the velocity vectors are all parallel and equal in length, and the remaining stream lines may be viewed as generalized homothetic transformations of S_1 in which the radii of similitude are the isobars and the center of similitude R moves along S_0.

Due to the arbitrary function $B(p)$ in (36), we may elect any hodograph curve h and still retain wide latitude in the choice of the initial stream line S_0. Fixing p_0, θ_0 in (34), it is possible to make S_0 coincide with any curve C passing through the origin of the physical plane and making an angle θ_0 with the x-axis, provided C has a finite radius of curvature R, e. g., is not a straight line. To see this we take

$$C:\quad x = \int_{\theta_0}^{\theta} R\cos\theta\, d\theta, \qquad y = \int_{\theta_0}^{\theta} R\sin\theta\, d\theta, \qquad R = R(\theta),$$

as parametric equation for C, and using (34) to introduce p as parameter on C; it follows from (36) that S_0 will coincide with C, provided we take $B = R(\theta(p))\theta'(p)$.

If we designate flows for which $\theta = \theta(p)$ as *isoclinic* and flows which have a one-dimensional hodograph as *Prandtl-Meyer flows*, we have seen that any irrotational, isoclinic flow is necessarily a Prandtl-Meyer flow in which the isobars are straight lines along which the velocity vectors are parallel and equal in length. Such flows have been studied in detail by various authors [6] for the special case of a polytropic gas.

There exist, however, rotational, isoclinic flows which are not Prandtl-Meyer flows. To study them, we return to (28) and examine it in its second rôle. From this standpoint we set

$$\theta = \theta(p), \tag{38}$$

where $\theta(p)$ is an arbitrarily chosen function, and regard (28) momentarily as an ordinary differential equation for q. If q_1, q_2 denote two linearly independent solutions of (28), the general solution is

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[6] See, for example, R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, pp. 259-293. Interscience Publishers, New York (1948).

$$q = a_1 q_1 + a_2 q_2, \tag{39}$$

where a_1, a_2 denote arbitrary constants, if (28) is regarded as an ordinary differential equation, but denote arbitrary functions of ψ when (28) is interpreted as a partial differential equation.

If we specify the arbitrary functions $a_1(\psi)$, $a_2(\psi)$ in (39), we obtain a definite Bernoulli function $q(p, \psi)$, corresponding to which the function $\theta(p)$, assigned in (38), yields a solution of (28). If we substitute these functions for q, θ in (30), (31), the second column of (31) yields a one-parameter family of hodograph curves

$$h: \quad u = a_1(\psi) u_1 + a_2(\psi) u_2, \qquad v = a_1(\psi) v_1 + a_2(\psi) v_2,$$

where

$$h_1: \quad u_1 = q_1 \cos\theta, \quad v_1 = q_1 \sin\theta; \qquad h_2: \quad u_2 = q_2 \cos\theta, \quad v_2 = q_2 \sin\theta,$$

represent two "base curves" in the hodograph plane which serve to determine the one parameter family of hodograph curves as indicated in Figure 4.

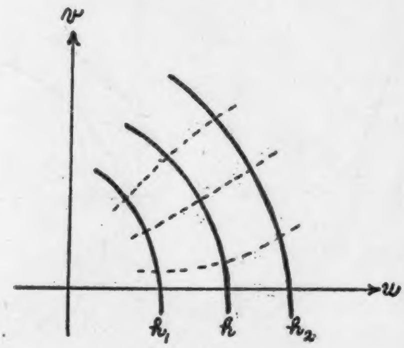

Figure 4.

It follows from (39) and Theorem 2 that our isoclinic flow will be irrotational if, and only if, a_1, a_2 are constant, i. e., if, and only if, h consists of a single curve. Consequently an isoclinic flow will be irrotational if, and only if, it is a Prandtl-Meyer flow.

According to the Munk-Prim Substitution Principle,[7] if u, v denote the velocity components for a given flow, the flow with velocity components λu, λv has the same stream lines and isobars as the given flow, provided $\lambda = \lambda(\psi)$. If two such flows are termed *equivalent*, it follows that all flows equivalent

[7] M. M. Munk and R. C. Prim, *On the multiplicity of steady gas flows having the same stream line pattern.* NOLM 9271, Naval Ordnance Laboratory (1947).

to a given isoclinic flow are isoclinic. Moreover an isoclinic flow will be equivalent to an irrotational flow if, and only if, the functions a_1, a_2 in (39) are linearly dependent. Consequently the necessary and sufficient condition for an isoclinic flow to be equivalent to an irrotational, Prandtl-Meyer flow is that a_1, a_2 be linearly dependent.

The isobars in the hodograph plane are represented by the dotted lines in Figure 4. To plot them p is fixed in h, and ψ allowed to vary.

The family

$$H: \quad U = A_1(\psi)u_1 + A_2(\psi)u_2, \qquad V = A_1(\psi)v_1 + A_2(\psi)v_2,$$

of curves in the hodograph plane, derived from h by setting

$$(40) \qquad A_1 = \int_{\psi_0}^{\psi} a_1(\psi)\,d\psi, \qquad A_2 = \int_{\psi_0}^{\psi} a_2(\psi)\,d\psi, \qquad \psi_0 = \text{const.}$$

enables one to draw the stream lines in the physical plane. Carrying out the

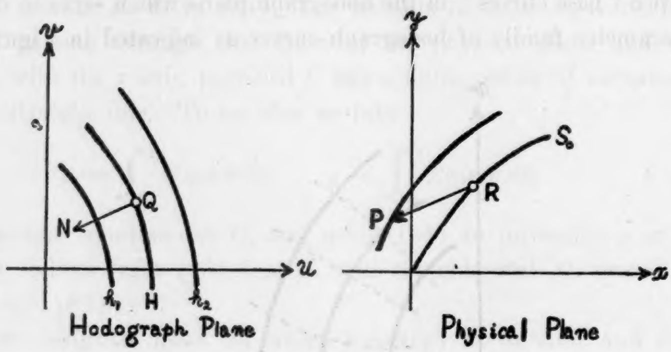

Figure 5.

integration of the line integrals in the first column of (31) along the broken line P_0AP in Figure 2, one obtains

$$(41) \qquad x = x_0(p) + V_p, \qquad y = y_0(p) - U_p.$$

Here

$$(42) \qquad x_0(p) = \int_{p_0}^{p} B\cos\theta\,dp, \qquad y_0(p) = \int_{p_0}^{p} B\sin\theta\,dp,$$

where θ is the function of p introduced in (38) and B is the arbitrary function of p in (30). From (40) both U_p, V_p vanish for $\psi = \psi_0$ so that

$$S_0: \quad x = x_0(p), \qquad y = y_0(p),$$

is the "initial stream line" $\psi = \psi_0$. To construct the other stream lines we refer to Figure 5.

Choosing a value of ψ, one draws the corresponding member H of the above family and selects a point R on S_0. Corresponding to R, we select a point Q on H so that p has the same value at both points, and erect the normal $\overrightarrow{QN}$ with components $(V_p, -U_p)$ to H at Q. It follows from (41) that the tip P of the vector $\overrightarrow{RP}$ drawn from R, equal in length and parallel to $\overrightarrow{QN}$, traces out the stream line for the selected value of ψ as R traverses S_0.

Because of the arbitrary function $B(p)$ in (42), once the Bernoulli function (39) has been fixed, it is still possible to take the initial stream line S_0 to be any curve through the origin of the physical plane, provided its angle of inclination there equals $\theta(p_0)$ and its radius of curvature is always finite.

An isoclinic flow is never subsonic, since $q_{pp} \geqq 0$ holds from (28).

Taking the Bernoulli function (39) for isoclinic flows, it follows from (19) that the density function necessarily has the form

$$\rho = -(a_1 q_1 + a_2 q_2)^{-1}(a_1 q'_1 + a_2 q'_2)^{-1},$$

where a_1, a_2 denote arbitrary functions of ψ and q_1, q_2 are linearly independent solutions of $q'' - \theta'^2 q = 0$.

Conversely, if we start with such a density function, (6) will permit solutions (14) for which $\theta = \theta(p)$, i. e., isoclinic flows are possible. To prove this, one substitutes for ρ in (15) and disposes the arbitrary functions $p_0(\psi)$, $q_0(\psi)$ so that the Bernoulli function is given by (39). Substituting for q, θ in (30), (31) one obtains solutions (14) of (6) of the desired type.

Let us assume that the entropy distribution function $S = S(\psi)$ can be inverted, $\psi = \psi(S)$ so that a_1, a_2 can be regarded as functions of S. If we designate these functions by σ_1^{-1}, $\sigma_2\sigma_1^{-1}$ we can state the following theorem.

THEOREM 5. *For a fluid to permit isoclinic flows for which $\theta = \theta(p)$ under the hypotheses of Theorem 1, it is necessary and sufficient that the equation of state has the form*

$$\rho = -\sigma_1^2(q_1 + \sigma_2 q_2)^{-1}(q'_1 + \sigma_2 q'_2)^{-1}, \qquad 1 = d/dp,$$

where q_1, q_2 denote linearly independent solutions of $q'' - \theta'^2 q = 0$, and σ_1, σ_2 are arbitrary functions of the entropy S, subject to the conditions $q > 0$, $\rho > 0$.

As an application of this theorem, consider the simplest case possible, $\theta = \text{const.}$ The stream lines are parallel straight lines and we may take $q_1 = -p$, $q_2 = 1$. Consequently the Bernoulli function must be $q = a_2 - a_1 p$, a linear function of p, and the equation of state must have the form $\rho = \sigma_1^2(\sigma_2 - p)^{-1}$. Naturally other forms for the equation of state are

to a given isoclinic flow are isoclinic. Moreover an isoclinic flow will be equivalent to an irrotational flow if, and only if, the functions a_1, a_2 in (39) are linearly dependent. Consequently the necessary and sufficient condition for an isoclinic flow to be equivalent to an irrotational, Prandtl-Meyer flow is that a_1, a_2 be linearly dependent.

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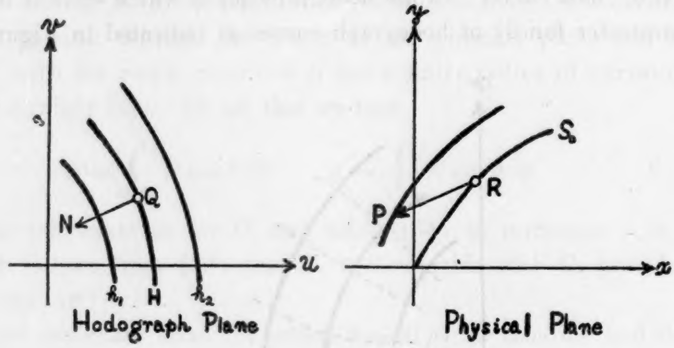

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$$\rho = -(a_1 q_1 + a_2 q_2)^{-1}(a_1 q'_1 + a_2 q'_2)^{-1},$$

where a_1, a_2 denote arbitrary functions of ψ and q_1, q_2 are linearly independent solutions of $q'' - \theta'^2 q = 0$.

Conversely, if we start with such a density function, (6) will permit solutions (14) for which $\theta = \theta(p)$, i. e., isoclinic flows are possible. To prove this, one substitutes for ρ in (15) and disposes the arbitrary functions $p_0(\psi)$, $q_0(\psi)$ so that the Bernoulli function is given by (39). Substituting for q, θ in (30), (31) one obtains solutions (14) of (6) of the desired type.

Let us assume that the entropy distribution function $S = S(\psi)$ can be inverted, $\psi = \psi(S)$ so that a_1, a_2 can be regarded as functions of S. If we designate these functions by σ_1^{-1}, $\sigma_2\sigma_1^{-1}$ we can state the following theorem.

THEOREM 5. *For a fluid to permit isoclinic flows for which $\theta = \theta(p)$ under the hypotheses of Theorem 1, it is necessary and sufficient that the equation of state has the form*

$$\rho = -\sigma_1^2(q_1 + \sigma_2 q_2)^{-1}(q'_1 + \sigma_2 q'_2)^{-1}, \qquad 1 = d/dp,$$

where q_1, q_2 denote linearly independent solutions of $q'' - \theta'^2 q = 0$, and σ_1, σ_2 are arbitrary functions of the entropy S, subject to the conditions $q > 0$, $\rho > 0$.

As an application of this theorem, consider the simplest case possible, $\theta = \text{const.}$ The stream lines are parallel straight lines and we may take $q_1 = -p$, $q_2 = 1$. Consequently the Bernoulli function must be $q = a_2 - a_1 p$, a linear function of p, and the equation of state must have the form $\rho = \sigma_1^2(\sigma_2 - p)^{-1}$. Naturally other forms for the equation of state are

possible if the gas flows on parallel straight lines under constant pressure; but if we presuppose the conditions of Theorem 1 (which require a change of pressure along a stream line) only gases having an equation of state of the above form are admissible.

The equation of state (12) for a polytropic gas is *separable*, i. e., it has the form $\rho = \Sigma(S)\Pi(p)$. For the equation of state in Theorem 5 to be separable it is clearly sufficient that $\sigma_2 = \text{const.}$ Under this assumption the functions a_1, a_2 in (39) are linearly dependent and the flow is equivalent to an irrotational Prandtl-Meyer flow.

Conversely, we shall show that this is the only case in which the equation of state is separable. In the proof we make use of the lemma below. This lemma is a generalization of a lemma frequently used in solving partial diffeıential equations by separation of variables which states that, for $n = 1$, the identity $\sum_{i=1}^{n} x_i(u)y_i(v) \equiv 1$ implies that $x_1(u)$ and $y_1(v)$ are reciprocal constants. For simplicity, and because it is adequate for our purpose, we restrict attention to the case $n = 3$, although no doubt the lemma can be extended to all positive integral values of n.

LEMMA 1. *In order that two curves*

$$C\colon \quad x_1 = x_1(u), \qquad x_2 = x_2(u), \qquad x_3 = x_3(u),$$
$$D\colon \quad y_1 = y_1(v), \qquad y_2 = y_2(v), \qquad y_3 = y_3(v),$$

satisfy the relation $x_1y_1 + x_2y_2 + x_3y_3 = 1$ identically in u, v, it is necessary and sufficient that one curve be a point and the other lie on the polar plane of this point with respect to the unit sphere, or that both curves lie on straight lines which are transforms of each other under polar reciprocation, i. e., either

(i) *C is a point (a_1, a_2, a_3) and D a curve on the plane*

$a_1y_1 + a_2y_2 + a_3y_3 = 1$, *or*

D is a point (b_1, b_2, b_3) and C a curve on the plane

$b_1x_1 + b_2x_2 + b_3x_3 = 1$,

or the curves C, D are presented parametrically by

$$\text{(ii)} \quad \begin{aligned} C\colon &\quad x_1 = a_1 + Ul_1,\ x_2 = a_2 + Ul_2,\ x_3 = a_3 + Ul_3, &\quad U = U(u), \\ D\colon &\quad y_1 = b_1 + Vm_1,\ y_2 = b_2 + Vm_2,\ y_3 = b_3 + Vm_3, &\quad V = V(v), \end{aligned}$$

where a_i, b_i, l_i, m_i denote constants satisfying the relations

$$a_1b_1 + a_2b_2 + a_3b_3 = 1, \qquad a_1m_1 + a_2m_2 + a_3m_3 = 0,$$
$$l_1b_1 + l_2b_2 + l_3b_3 = 0, \qquad l_1m_1 + l_2m_2 + l_3m_3 = 0.$$

For the proof of this lemma we employ a geometrical argument. To a fixed point $B(b_1, b_2, b_3)$ of D there corresponds the polar plane $b_1x_1 + b_2x_2 + b_3x_3 = 1$ of B with respect to the unit sphere. Since C, D satisfy the relation $x_1y_1 + x_2y_2 + x_3y_3 = 1$ identically in u, v, the curve C must lie in this polar plane. If D consists of a single point B, the identity is fulfilled if, and only if, C is a curve in the polar plane of B and the case comes under alternative (i). If D is not a single point the polar planes corresponding to the points of D constitute a one-parameter family of planes. For the identity to be fulfilled, it is necessary and sufficient that C be in the set S of points common to these planes. For the set S to be not vacuous, it is necessary and sufficient that D be a straight line or a plane curve. In the first case the polar planes are co-axial to yield alternative (ii), and in the latter case they have a single point (a_1, a_2, a_3) in common, as in alternative (i).

Returning to the proof of the necessity of the condition $\sigma_2 = \text{const.}$, in order that the equation of state be separable, it is clear that the product $(q_1 + \sigma_2 q_2)(q'_1 + \sigma_2 q'_2)$ must be separable, consequently the identity in Lemma 1 must be satisfied by the curves

$$C:\quad x_1 = -(q_1q_2)'/q_1q'_1 = x_1(p),\quad x_2 = -q_2q_2'/q_1q'_1 = x_2(p),\quad x_3 = x_3(p),$$
$$D:\quad y_1 = \sigma_2 = y_1(S),\qquad y_2 = \sigma_2{}^2 = y_2(S),\qquad y_3 = y_3(S).$$

If C were a point (a_1, a_2, a_3), it would follow that

$$q_1q_2 = -\tfrac{1}{2}a_1q_1{}^2 + c_1,\qquad q_2{}^2 = -a_2q_1{}^2 + c_2,\qquad c_1 = \text{const.},\ c_2 = \text{const.},$$

and hence that q_1, q_2 are both constant or else linearly dependent, neither of which is possible, since q_1, q_2 designate linearly independent solutions of $q'' - \theta'^2 q = 0$. Thus C cannot be a point. The projection of D on the plane $y_3 = 0$ is the parabola $y_1{}^2 = y_2$, so D cannot contain a straight line segment. Consequently D is a point, and σ_2 is a constant as stated.

These results lead to the following theorem.

THEOREM 6. *The rotational, isoclinic flows of a gas with a separable equation of state subject to the conditions in Theorem 1 are equivalent to irrotational, Prandtl-Meyer flows under the Munk-Prim Substitution Principle.*

6. The case $\theta = \theta(\psi)$. Equation (26), like (28), plays two rôles, depending on whether q, or θ, is taken to be an assigned function of p, ψ. Taking (26) in its second rôle, we prescribe $\theta = \theta(\psi)$ and take q for unknown function. We suppose $\theta' \neq 0$, so that the stream lines lie on a family of non-parallel straight lines and term such flows *rectilinear* flows. For rectilinear flows (26) reduces to $(q_{pp}/\theta_\psi)_\psi = 0$, the general solution of which

$$(43)\qquad q = \dot{\theta}[P(p) + \Psi_1(\psi)p + \Psi_2(\psi)],\qquad \dot{\theta} \neq 0,\qquad (\cdot = d/d\psi),$$

involves three arbitrary functions $P(p)$, $\Psi_1(\psi)$, $\Psi_2(\psi)$. If we specify these arbitrary functions, we obtain a definite Bernoulli function $q(p,\psi)$ corresponding to which the prescribed function $\theta(\psi)$ yields a solution of (26), and, on substituting for q, θ in (27), we obtain a solution (14) of (6) for which the density function is given by

$$(44)\qquad \rho = -\dot{\theta}^{-2}(P' + \Psi_1)^{-1}(P + \Psi_1 p + \Psi_2)^{-1}.$$

Conversely, if we start with a density function of this type, (6) will permit solutions (14) for which $\theta = \theta(\psi)$, i. e., rectilinear flows are possible. To obtain them, one substitutes from (44) into (15) and adjusts the arbitrary functions $p_0(\psi)$, $q_0(\psi)$ so that the Bernoulli function is given by (43). The desired solution (14) of (6) is then obtained by substituting for q, θ in (27).

Introducing the entropy S as in Section **5**, we obtain the theorem in view of (23).

THEOREM 7. *For a fluid to permit rectilinear flows under the hypotheses of Theorem* 1, *it is necessary and sufficient that the equation of state has the form*

$$(45)\qquad \rho = -\sigma^2(P' + \sigma_1)^{-1}(P + \sigma_1 p + \sigma_2)^{-1}, \qquad ' = d/dp,$$

where P is any non-linear function of the pressure p and σ, σ_1, σ_2 denote arbitrary functions of the entropy S, subject to the conditions $q > 0$, $\rho > 0$.

Further insight into rectilinear flows is given by the following theorem.

THEOREM 8. *Under the hypotheses of Theorem* 1, *the necessary and sufficient conditions for the isobars to be orthogonal trajectories of the stream lines is that the stream lines lie on straight lines.*

If we let K denote the curvature of a stream line this theorem is an immediate consequence of the relation $x_p x_\psi + y_p y_\psi = qK(ds/dp)^2$, derived by substituting for x_p, y_p from (22) and for x_ψ, y_ψ from (6) with the help of (21).

Consequently if the stream lines lie on straight lines L the isobars are the involutes I of the evolute E enveloped by the straight lines.

To confirm and elaborate upon this result, we substitute for q from (43) and put $\theta = \theta(\psi)$ in (27) to obtain

$$x = \int \Psi_1 \sin\theta d\theta - P' \cos\theta, \qquad y = -\int \Psi_1 \cos\theta d\theta - P' \sin\theta.$$

and this, upon integration by parts, yields

$$(46)\qquad \begin{aligned} x &= \int (\dot{\Psi}_1/\dot{\theta}) \cos\theta d\theta - (P' + \Psi_1)\cos\theta, \\ y &= \int (\dot{\Psi}_1/\dot{\theta}) \sin\theta d\theta - (P' + \Psi_1)\sin\theta. \end{aligned}$$

If we let R denote the radius of curvature of E and τ be the arc length of E measured from some fixed point P_0 on E, the equations of the involutes of E are

$$(47) \qquad x = \int R \cos\theta d\theta + (a - \tau)\cos\theta, \qquad y = \int R \sin\theta d\theta + (a - \tau)\sin\theta,$$

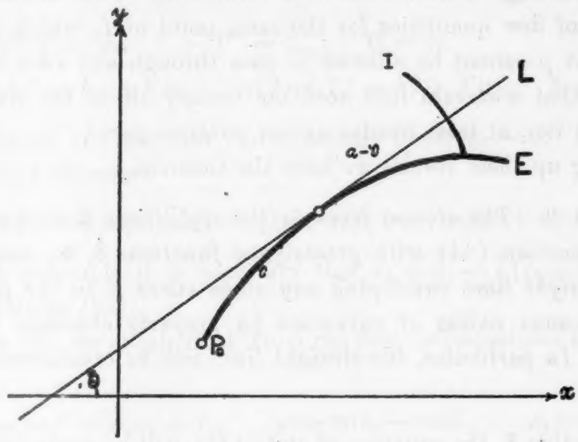

Figure 6

in which a specific involute is obtained by assigning a definite value to the constant a. Comparing (47) with (46), we set

$$(48) \qquad R = \dot{\Psi}_1/\dot{\theta}, \qquad \tau = \psi_1, \qquad a = -P'.$$

Consequently the stream lines in (46) lie on straight lines enveloping a curve E whose intrinsic equation $R = R(\tau)$ is obtained by eliminating ψ from the first two equations in (48). It follows that E is uniquely determined up to rotations and translations by prescribing the arbitrary function $\Psi_1(\psi)$. The third equation in (48) determines the pressure to be assigned to each involute I in its rôle as an isobar, provided, of course that $P' \neq$ const. From (43), (44) it is clear that neither the lines of constant speed (isovels) nor the lines of constant density (isopycnics) are necessarily the same as the isobars, as is always the case in irrotational flows.

The arbitrary function $\Psi_1(\psi)$ may be chosen so that E is any curve in the physical plane with a continuous radius of curvature, for if $R = R(\tau)$ is the intrinsic equation of E, it is sufficient, from (48), to take any solution of the differential equation $\dot{\Psi}_1 = \theta R(\Psi_1)$ for Ψ_1.

Concurrent straight lines are characterized by $R = 0$, and therefore, from (48), by the condition $\Psi_1 =$ const.

Referring to Figure 6, if we denote the distance $a - \tau$ measured along L from its point of tangency with E to the isobar I by s, it follows from (48), (43) that $s = - q_p/\dot{\theta}$. Consequently if p passes through a value p^* at which q_{pp} changes sign, the flow changes from subsonic to supersonic or vice versa and s passes through a maximum or a minimum. This would lead to two different sets of flow quantities for the same point of L, which is impossible. It follows that p cannot be allowed to pass through any such value p^* and consequently that a stream line need not occupy all of the straight line L upon which it lies, at least insofar as our analysis goes.[8]

Summing up these results we have the theorem.

THEOREM 9. *The stream lines in the rectilinear flow of a gas having the density function* (44) *with preassigned functions* $\dot{\theta}$, Ψ_2 *may be made to lie on the straight lines enveloping any given curve* E *in the physical plane with a continuous radius of curvature by properly choosing the arbitrary function* Ψ_1. *In particular, the straight lines will be concurrent if, and only if* $\Psi_1 = const.$

As in Section **5**, the equation of state (45) will be separable if, and only if, the product $(P' + \sigma_1)(P + \sigma_1 p + \sigma_2)$ is separable.

LEMMA 2. *The product* $(P' + \sigma_1)(P + \sigma_1 p + \sigma_2)$ *is separable, if, and only if, one of the following alternatives holds.*

(i) $\sigma_1 = c_1, \quad P = - c_1 p + c_2;$ (ii) $\sigma_1 = c_1, \quad \sigma_2 = c_2;$

(iii) $P = c_1 p + c_2, \quad c_2 + \sigma_2 = k(c_1 + \sigma_1),$

where c_1, c_2, k *denote constants.*

It is a simple matter to verify that any one of these alternatives is sufficient.

To prove that the three alternatives exhaust the possibilities, we exclude (i) and show that either (ii) or (iii) must hold.

Under the hypothesis $\sigma_1 = c_1$ the product $(P' + c_1)(P + c_1 p + \sigma_2)$ must be separable and, since (i) is excluded, the factor $P + c_1 p + \sigma_2$ must be separable, which can only be the case if $\sigma_2 = \text{const.}$, i. e., if (ii) holds.

If, on the other hand, $\dot{\sigma}_1 \neq 0$, we differentiate the identity

$$(48) \qquad (P' + \sigma_1)(P + \sigma_1 p + \sigma_2) = PP' + (pP')\sigma_1 + P'\sigma_2 + p\sigma_1^2 + \sigma_1\sigma_2,$$

[8] S. Bergman, *The hodograph method in the theory of compressible fluid.* Brown University Publication, Providence (1941), in which the flow of a polytropic gas for a compressible source or sink is investigated and the analysis is restricted to the exterior of a "limit circle."

partially with respect to p and S, and, since the result must be separable, we may write $(pP)''\dot{\sigma}_1 + P''\dot{\sigma}_2 + 2\sigma_1\dot{\sigma}_1 = \Pi\Sigma$, $\Pi = \Pi(p)$, $\Sigma = \Sigma(S)$, which, on dividing by $2\sigma_1\dot{\sigma}_1$ may be replaced by $-(pP)''(2\sigma_1)^{-1} - P''\dot{\sigma}_2(2\sigma_1\dot{\sigma}_1)^{-1} + \Pi\Sigma_1 = 1$, $\Sigma_1 = (2\sigma_1\dot{\sigma}_1)^{-1}\Sigma$, to which we apply Lemma 1. Since $\sigma_1 \neq$ const., either one of

(a) $\quad -(pP)'' = a_1, \qquad -P'' = a_2, \qquad \Pi = a_3,$

(b) $\quad (2\sigma_1)^{-1} = b_1 + Vm_1, \qquad \dot{\sigma}_2(2\sigma_1\dot{\sigma}_1)^{-1} = b_2 + Vm_2, \qquad \Sigma_1 = b_3 + Vm_3,$

must hold. Under (a) the first two equations imply $a_2 = 0$. Consequently $P = c_1 p + c_2$ and therefore

$$(P' + \sigma_1)(P + \sigma_1 p + \sigma_2) = (c_1 + \sigma_1)[(c_1 + \sigma_1)p + c_2 + \sigma_2].$$

For this to be separable it is necessary that $c_2 + \sigma_2 = k(c_1 + \sigma_1)$, and we arrive at alternative (iii).

Taking up (b), we eliminate V from the first two equations and integrate to obtain

(49) $\quad m_1\sigma_2 = \mu\sigma_1^2 + m_2\sigma_1 + c_1, \qquad \mu = m_1 b_2 - m_2 b_1, \qquad c_1 = \text{const.}$

where $m_1 \neq 0$, since if $m_1 = 0$, the first equation in (b) implies $\sigma_1 =$ const. If we use (49) to eliminate σ_2 from (48), the identity becomes

(50) $$m_1(P' + \sigma_1)(P + \sigma_1 p + \sigma_2) = (c_1 + m_1 P)P' + \{c_1 + [(m_1 p + m_2)P]'\}\sigma_1 + (m_1 p + m_2 + \mu P')\sigma_1^2 + \mu\sigma_1^3.$$

Differentiating partially with respect to S, we obtain a necessary condition for separability which, provided $\mu \neq 0$, may be identified with the identity in Lemma 1 for the curves

$$C:\quad x_1 = -c_1 - [(m_1 p + m_2)P]', \qquad x_2 = -(m_1 p + m_2 + \mu P'), \qquad x_3 = x_3(p),$$

$$D:\quad y_1 = (3\mu\sigma_1^2)^{-1}, \qquad y_2 = 2(3\mu\sigma_1)^{-1}, \qquad y_3 = y_3(S).$$

Clearly D cannot be a point or straight line segment, so C must be a point. Now x_1, x_2 can both be constant only if $m_1 = 0$, and since actually $m_1 \neq 0$, the assumption $\mu \neq 0$ leads to a contradiction and must be rejected.

For $\mu = 0$, the condition for separability of the product in (50) assumes the form of the identity in Lemma 1 for the curves

$$C:\quad x_1 = -(c_1 + m_1 P)P'/(m_1 p + m_2),$$
$$x_2 = -\{c_1 + [(m_1 p + m_2)P]'\}/(m_1 p + m_2), \quad x_3 = x_3(p),$$

$$D:\quad y_1 = \sigma_1^{-2} \qquad y_2 = \sigma_1^{-1}, \qquad y_3 = y_3(S).$$

As above D cannot be a point or straight line segment, so C must be a point. This time, however, for x_1, x_2 to reduce to constants it is necessary that P be a linear function of p and proceeding as in (a), we are led to alternative (iii).

Lemma 2 is used to prove the following theorem.

THEOREM 10. *Under the hypotheses of Theorem* 1 *the stream lines in the rectilinear flow of a gas with a separable equation of state must lie on concurrent straight lines and the flow upon them is equivalent to an irrotational flow under the Munk-Prim Substitution Principle.*

For a rectilinear flow with the Bernoulli function (43) it is easy to verify that (23) reduces to $dp/ds = -1/P''$, and, if the pressure gradient is to remain finite for a gas with a separable equation of state, alternatives (i), (iii) in Lemma 2 are ruled out. Under alternative (ii) both σ_1, σ_2 are constant in (45), the straight lines are concurrent by Theorem 9, and the flow upon them equivalent, from (43), to an irrotational flow under the Munk-Prim Substitution Principle.

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ON THE ASYMPTOTIC SHAPE OF THE CAVITY BEHIND AN AXIALLY SYMMETRIC NOSE MOVING THROUGH AN IDEAL FLUID.*

By Francis Scheid.

Introduction. This paper continues and in a sense completes that of Levinson [1] in which it is shown that the application of Green's theorem to the problem described in the title leads to a singular, non-linear integro-differential equation which can be written in the form [2]

$$(3.10) \qquad \phi(x, f(x)) = I_1 + I_2 + I_3 + J_1 + J_2 + J_3 + \text{const.} + O(1/x),$$

where x is distance from the tip of the nose measured along the axis of symmetry, $r = f(x)$ is the equation of the free boundary, ϕ is a harmonic function such that $V + \phi_x$ and ϕ_r are the x and r components of the fluid velocity, V being the velocity at infinity, and the remaining symbols denote surface integrals whose exact form will appear as we proceed.

The problem of cavity shapes in two dimensions has received considerable attention, the principle technique being that of conformal mapping. However, in the three-dimensional case no general method is known. For the problem at hand Levinson's result is

$$(1.4) \qquad f(x) = (cx^{1/2}/\log^{1/4} x)\,[1 - (\log\log x)/(8\log x) + O(1/\log x)],$$

where x is large and c constant. In obtaining this result certain assumptions are made involving the smoothness of $f(x)$ leading to an asymptotic appraisal of the integrals in (3.10) which then gives a weighted mean of $f(x)$. (1.4) is then a consequence of what amounts to a Tauberian theorem.

Here we make the additional assumption that the unknown quantity in (1.4) has a derivative which is $O(1/x\log^2 x)$ and find by differentiation

$$(1) \qquad f'(x) = (f(x)/2x)\,[1 - 1/(2\log x) + (\log\log x)/(4\log^2 x) + O(1/\log^2 x)].$$

In Part I which follows, the integrals in (3.10) are reduced to a form which makes their reappraisal a matter of substitution and computation. Details are provided so that in Part II we may extend (1.4) to

* Received June 9, 1949.

$$f(x) = (cx^{1/2}/\log^{1/4} x)[1 - (\log\log x)/(8\log x) + (1/4 + \tfrac{1}{2}\log(\tfrac{1}{2}c))/\log x + (5/128)(\log\log x/\log x)^2 - (7/32 + (5/16)\log\tfrac{1}{2}c)(\log\log x/\log^2 x) + O(1/\log^2 x)].$$

The nature of the computations necessary to continue this series makes it apparent that the shape of the nose, except for the general assumptions already made, has no effect upon the asymptotic shape of the cavity wall.

Other equations obtained by Levinson which we shall use are:

(2.17) $\phi_n = (f'(x))(1 + f'^2(x))^{-\frac{1}{2}}$;

(3.5) $f(x+y) < 2f(x)$, if $|y| \leq \tfrac{1}{2}x$;

(3.15) $\displaystyle\iint_{w_2} \cos\psi/R^2\, dS = -2\pi + O(f^2(x)/x^2)$;

(6.5) $\phi(x, f(x)) = C_2 + \tfrac{1}{2}B(x) + O(x^{-\frac{1}{2}})$;

(6.6) $J_1 = O(x^{-1+\epsilon})$;

(6.7) $J_3 = O(x^{-1+\epsilon})$;

(6.15) $\displaystyle B(x) = \int_a^x f'^2(t)\,dt = \text{const.} + E_1 + (1/4)\int_a^x f^2(t)/t^2\,dt,$

where $\displaystyle E_1 = -\int_x^\infty [f'^2(t) - 1/4\, f^2(t)/t^2]\,dt$;

(6.17) $\displaystyle\int_a^x f(t)f'(t)/(t+1)\,dt = \text{const.} + F_1 + \tfrac{1}{2}\int_a^x f^2(t)/t^2\,dt,$

where $\displaystyle F_1 = -\int_x^\infty [f(t)f'(t)/(t+1) - \tfrac{1}{2}f^2(t)/t^2]\,dt$;

(6.19) $\displaystyle R(y) = \int_{\log a}^y H(n)/n^{\frac{1}{2}}\,dn - 2y^{\frac{1}{2}}H(y)$;

(6.20) $\displaystyle H(y) = c^2 - \tfrac{1}{2}R(y)/y^{\frac{1}{2}} + (1/4)\int_y^\infty R(n)/n^{3/2}\,dn.$

More recently M. I. Gurevich has obtained the same result as Levinson, but in an entirely different manner. Gurevich's paper appears in *Prikladnaia Matematika y Mekhanika,* vol. 11, No. 1, and is titled "Flow past an Axi-symmetric Semi-Body of Finite Drag."

Part I. Reappraisal of the Integrals.

We consider first $I_1 = -\frac{1}{2\pi}\displaystyle\iint_{w_1}(1/R - 1/(\xi+1))\phi_n dS$. The second term gives, using (2.17), (1), and (1.4),

$$\left(\int_a^x - \int_{3x/4}^x\right)[f(\xi)f'(\xi)/(\xi+1)]d\xi + \left(\int_a^\infty - \int_{3x/4}^\infty\right)[f(\xi)O(f'^3(\xi))/(\xi+1)]d\xi,$$

where the last term is of the form const. $+\, O(1/x)$. The first term gives

$$-\tfrac{1}{2\pi}\iint_{w_1} f'(\xi)/R\, dS - \tfrac{1}{2\pi}\iint_{w_1} O(f'^3(\xi))/R\, dS = Q_1 + Q_2.$$

Now $|Q_2| \leq 4/x \int_a^x f(\xi)O(f'^3(\xi))d\xi = O(\operatorname{Log} x/x)$.

For Q_1 we have $1/R = [(x-\xi)^2 + r^2 + \rho^2 - 2r\rho\cos(\theta-\lambda)]^{-\frac12}$

$$= [1 + O(f^2(x)/x^2)]^{-\frac12}/(x-\xi) = 1/(x-\xi) + O(f^2(x)/x^3)$$

and so

$$Q_1 = -\tfrac{1}{2\pi}\iint_{w_1} f'(\xi)/(x-\xi)dS + \iint_{w_1} O(f^2(x)/x^3)f'(\xi)dS$$

$= Q_3 + Q_4$. But $|Q_4| \leq 2\pi f^2(x)/x^3$. const. $\int_a^{3x/4} f(\xi)f'(\xi)d\xi$

$$< \text{const.}\, f^3(x)/x^3 \int_a^x f'(\xi)d\xi = O(1/x).$$

Next we have

$$Q_3 = -\int_a^{3x/4} f(\xi)f'(\xi)/(x-\xi)d\xi = \left(-\int_a^b - \int_b^{3x/4}\right)[f(\xi)f'(\xi)/(x-\xi)]d\xi$$

$= Q_5 + Q_6$, where $b = x/\log^n x$. Using (1.4) we find

$$|Q_5| \leq [1/(x-b)]f^2(b) \leq \text{const.}\, b/(\log^{\frac12} b)(x-b) = O(\log^{-n-\frac12} x).$$

We choose $n \geq 1$. Once again using (1) we have

$$Q_6 = -\int_b^{3x/4} [1 + O(1/\log x)]f^2(\xi)/2\xi(x-\xi)d\xi.$$

Since $\int_b^{3x/4} d\xi/(x-\xi)$ is bounded for large x, the error term contributes $O(\log^{-3/2} x)$. Thus

$$Q_6 = -\int_b^{3x/4} f^2(x)/2x(x-\xi)d\xi - \int_b^{3x/4} [f^2(\xi)/2\xi - f^2(x)/2x]/(x-\xi)d\xi + O(\log^{-3/2} x)$$

$$= -f^2(x)(\log 4)/2x - \int_b^{3x/4} + \int_x^\xi [f(t)f'(t)/t - f^2(t)/2t^2]dt\, d\xi/(x-\xi) + O(\log^{-3/2} x)$$

$$= -f^2(x)(\log 4)/2x + O(\log^{-3/2} x)$$

$$-\left(\int_x^{3x/4}\int_b^{3x/4} + \int_{3x/4}^{b}\int_b^{t}\right)[f(t)f'(t)/t - f^2(t)/2t^2]\,d\xi/(x-\xi)\,dt$$

$$= -f^2(x)(\log 4)/2x - Q_7 - Q_8 + O(\log^{-3/2} x).$$

In Q_7 the integrations can be performed so that

$$Q_7 = \int_x^{3x/4}(\log 4 + O(\log^{-n} x))\,d/dt(f^2(t)/2t)\,dt$$

$$< \text{const.}\ [f^2(3x/4)/(3x/4) - f^2(x)/x], \text{ and now using (1.4)}$$

$$< \text{const.}\ [(1/(\log^{\frac12} 3x/4) - 1/(\log^{\frac12} x))$$

$$-((\log\log 3x/4)/4(\log 3x/4)^{3/2} - (\log\log x)/4\log^{3/2} x)$$

$$+ O(\log^{-3/2} x)]$$

$= O(\log^{-3/2} x)$, where the following identities are used:

(A) $\log\log 3x/4 = \log\log x + O(\log^{-1} x)$

(B) $\log^{-1/2} 3x/4 = \log^{-1/2} x + O(\log^{-3/2} x)$

(C) $\log^{-3/2} 3x/4 = \log^{-3/2} x + O(\log^{-5/2} x)$.

In Q_8 carrying out the ξ integration gives

$$Q_8 = \int_{3x/4}^{b}[f(t)f'(t)/t - f^2(t)/2t^2]\log[(x-b)/(x-t)]\,dt, \text{ which by (1)}$$

$$= \int_{3x/4}^{b}(f^2(t)/4t^2\log t)[-1 + (\log\log t/(2\log t))$$

$$+ O(1/\log t)]\log[(x-b)/(x-t)]\,dt.$$

Over this range of integration $\log[(x-b)/(x-t)]$ is bounded; moreover, by (1.4),

$$f^2(t)/(t\log t) < k/\log^{3/2} t < k/\log^{3/2} b = O(\log^{-3/2} x),$$

k being a constant. Accordingly in Q_8, the last two terms in brackets contribute $O(\log^{-3/2} x)$. Noting also that $\log(x-b) = \log x + O(\log^{-n} x)$ we have

$$Q_8 = \int_b^{3x/4}(f^2(t)/4t^2\log t)(\log x/x - t)\,dt + O(\log^{-3/2} x).$$

Now

$$(f^2(t)/t\log t) - (f^2(x)/x\log x)$$

$$< f^2(x\log^{-n} x)/[x\log^{-n} x\log(x\log^{-n} x)] - f^2(x)/x\log x$$

$$< \text{const.}(\log^{-3/2} b - \log^{-3/2} x) + O(\log^{-2} x) = O(\log^{-2} x),$$

where we have used the identity

$$\text{(D)} \quad (\log x - n \log\log x)^{-3/2} = \log^{-3/2} x + O(\log^{-2} x).$$

So replacing $f^2(t)/(t \log t)$ by $f^2(x)/(x \log x)$ in Q_8 gives an error of

$$O(\log^{-2} x) \int_b^{3x/4} dt/t = O(\log^{-3/2} x),$$

the logarithm in the integrand being bounded as before. But now

$$Q_8 = (f^2(x)/(4x \log x)) \int_b^{3x/4} \log (x/t(x-t))\, dt + O(\log^{-3/2} x),$$

and since $\log x - \log (x-t) < kt/x$, the remaining integral is bounded, so that $Q_8 = O(\log^{-3/2} x)$.

In summary then,

$$I_1 = \left(\int_a^x - \int_{3x/4}^x\right) f(\xi)f'(\xi)/(\xi+1)\, d\xi - (\log 4) f^2(x)/2x \\ + \text{const.} + O(\log^{-3/2} x).$$

Next we consider $I_2 = -\frac{1}{2\pi} \iint_{w_2} [1/R - 1/(\xi+1)]\phi_n\, dS$. Using (2.17), (1) and (1.4), the second term becomes

$$\left(\int_{3x/4}^x + \int_x^{5x/4}\right) f(\xi)f'(\xi)/(\xi+1)\, d\xi + O(1/x).$$

The first of these integrals occurs with the opposite sign in I_1 and for the second, using (1) and (1.4) we find

$$|f(x)f'(x) - f(\xi)f'(\xi)| < f(x)f'(x) - f(5x/4)f'(5x/4) \\ = \tfrac{1}{2}c^2[1/\log^{\frac{1}{2}} x - 1/\log^{\frac{1}{2}}(5x/4) - (\log\log x)/(4 \log^{3/2} x) \\ + (\log\log 5x/4)/(4 \log^{3/2} 5x/4)] + O(\log^{-3/2} x)$$

$= O(\log^{-3/2} x)$ just as before, using identities almost the same as (A), (B), and (C). Accordingly, since $\int_x^{5x/4} d\xi/(\xi+1) = \log 5/4 + O(1/x)$, $f(\xi)f'(\xi)$ may be replaced by $f(x)f'(x)$, introducing an error of $O(\log^{-3/2} x)$. Hence

$$\int_x^{5x/4} f(\xi)f'(\xi)/(\xi+1)\, d\xi = f(x)f'(x) \log 5/4 + O(\log^{-3/2} x),$$

$= (f^2(x)/2x)(\log 5/4) + O(\log^{-3/2} x)$, by (1). The second term in I_2 therefore contributes

$$\int_{3x/4}^{x} f(\xi)f'(\xi)/(\xi+1)\,d\xi + (f^2(x)/2x)(\log 5/4) + O(\log^{-3/2} x).$$

The first term in I_2 is $-\frac{1}{2\pi}\int\int_{w_2}(1/R)\phi_n\,dS$

$$= -\frac{1}{2\pi}\int\int_{w_2} f'(\xi)(1+f'^2(\xi))^{-\frac{1}{2}}[(x-\xi)^2+(r+\rho)^2$$
$$-2r\rho(1+\cos(\theta-\lambda))]^{-\frac{1}{2}}\,dS$$

$$= -\frac{1}{2\pi}\int_{3x/4}^{5x/4} f(\xi)f'(\xi)(1+f'^2(\xi))^{-\frac{1}{2}}$$
$$\int_0^{\frac{1}{2}\pi} 4[(x-\xi)^2+(r+\rho)^2-4r\rho\cos^2\lambda]^{-\frac{1}{2}}\,d\lambda\,d\xi$$

$$= -\frac{1}{2\pi}\int_{3x/4}^{5x/4} f(\xi)f'(\xi)(1+f'^2(\xi))^{-\frac{1}{2}}$$
$$\int_0^{\frac{1}{2}\pi} 4(4r\rho)^{-\frac{1}{2}}(\alpha^2+\sin^2\lambda)^{-\frac{1}{2}}\,d\lambda\,d\xi,$$

where we have used (2.17) and made a change of variable in the inner integral, and where

$$\alpha^2 = ((x-\xi)^2+(r+\rho)^2-4r\rho)/4r\rho = [(x-\xi)^2+(r-\rho)^2]/4r\rho$$
$$= ((x-\xi)^2/4r\rho)(1+(r-\rho)^2/(x-\xi)^2).$$

Now whether $\xi > x$ or $\xi < x$, $(r-\rho)^2/(x-\xi)^2 < f'^2(3x/4)$, so that

$$\alpha^2 = ((x-\xi)^2/4r\rho)(1+O(f^2(x)/x^2)). \tag{2}$$

It is convenient to split this last integral into the following pieces:

R_1 = part for which $x_1 \leq \xi \leq x_2$;

R_2 = part for which $x_2 \leq \xi \leq x_4$ or $x_3 \leq \xi \leq x_1$;

R_3 = part for which $x_4 \leq \xi \leq 5x/4$ or $3x/4 \leq \xi \leq x_3$,

where x_1, x_2, x_3 and x_4 denote respectively $x-\delta_n$, $x+\delta_n$, $x-x^{3/5}$, and $x+x^{3/5}$.

To treat R_1, we note that $\int_0^{\frac{1}{2}\pi}[(\alpha^2+\sin^2\lambda)^{-\frac{1}{2}}-(\alpha^2+\lambda^2)^{-\frac{1}{2}}]\,d\lambda$

$$= \int_0^{\frac{1}{2}\pi}(\lambda^2-\sin^2\lambda)(\alpha^2+\sin^2\lambda)^{-\frac{1}{2}}(\alpha^2+\lambda^2)^{-\frac{1}{2}}/[(\alpha^2+\sin^2\lambda)^{\frac{1}{2}}$$
$$+(\alpha^2+\lambda^2)^{\frac{1}{2}}]\,d\lambda$$

$$< \int_0^{\frac{1}{2}\pi}\text{const.}\,\lambda\,d\lambda = \text{const.}$$

Consequently, replacing $(\alpha^2+\sin^2\lambda)^{\frac{1}{2}}$ by $(\alpha^2+\lambda^2)^{\frac{1}{2}}$ in R_1 introduces an error of $<$ const. $\delta_n f(x)/x$ (by (1) and (3.5)). This error will be $O(\log^{-n} x)$ if we choose

$$\delta_n = b/f(x). \tag{3}$$

Hence

$$R_1 = -\frac{1}{\pi}\int_{x_1}^{x_2} f(\xi)f'(\xi)[1+f'^2(\xi)]^{-\frac{1}{2}}(r\rho)^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}(\alpha^2+\lambda^2)^{-\frac{1}{2}}d\lambda\, d\xi + O(\log^{-n} x),$$

with δ_n as in (3). But the λ integration can now be performed and

$$R_1 = -\frac{1}{\pi}\int_{x_1}^{x_2} f(\xi)f'(\xi)[1+f'^2(\xi)]^{-\frac{1}{2}}(r\rho)^{-\frac{1}{2}}\log\{[\tfrac{1}{2}\pi + (\alpha^2+\pi^2/4)^{\frac{1}{2}}]/|\alpha|\}d\xi + O(\log^{-n} x).$$

Now $\log\{[\frac{1}{2}\pi+(\alpha^2+\pi^2/4)^{\frac{1}{2}}]/|\alpha|\} = \log(\pi/|\alpha|) + O(\alpha^2)$, and observing that by (2), $\alpha^2 = O((x-\xi)^2/r\rho) = O(\delta^2{}_n/r^2) = O(\log^{1-2n} x)$, we find using (3.5) that the term $O(\alpha^2)$ in R_1 contributes

$$< \text{const.}\ (\log^{1-2n} x)[f(x_2)-f(x_1)]$$

$$< \text{const.}\ (\log^{1-2n} x)\delta_n f'(x_1) \quad \text{(by the mean-value theorem)}$$

$$= O((\log^{1-2n} x)\delta_n f(x)/x_1) = O(\log^{1-3n} x) \text{ by (1) and (3).}$$

The same argument shows that the $\log \pi$ term contributes $O(\log^{-n} x)$ so that

$$R_1 = \frac{1}{\pi}\int_{x_1}^{x_2} f(\xi)f'(\xi)[1+f'^2(\xi)]^{-\frac{1}{2}}(r\rho)^{-\frac{1}{2}}(\log|\alpha|)d\xi + O(\log^{-n} x).$$

By (2),

$$\log|\alpha| = \tfrac{1}{2}\log\alpha^2 = \tfrac{1}{2}\log\{[(x-\xi)^2/4r\rho][1+O(f^2(x)/x^2)]\}$$

$$= \tfrac{1}{2}\log((x-\xi)^2/4r\rho) + O(1/x).$$

As above the term $O(1/x)$ contributes $O(1/x)$; furthermore by (3.5) and (1.4), $|\log 4r\rho| < \log 8r^2 = O(\log r) = O(\log x)$, so again as before this term contributes $O(\log^{1-n} x)$ and

$$R_1 = \frac{1}{\pi}\int_{x_1}^{x_2} f(\xi)f'(\xi)[1+f'^2(\xi)]^{-\frac{1}{2}}(r\rho)^{-\frac{1}{2}}(\log|x-\xi|)d\xi + O(\log^{1-n} x).$$

Now $\displaystyle\int_{x_1}^{x_2}\log|x-\xi|\,d\xi = 2\int_0^{\delta_n}\log p\,dp = 2(\delta_n\log\delta_n - \delta_n) < \text{const.}\ \delta_n\log x,$

so that $|R_1| \leq$ const. $\delta_n \log xf(x)/x + O(\log^{1-n} x) = O(\log^{1-n} x)$.

We next turn to

$$R_2 = -\frac{1}{2\pi}\left(\int_{x_3}^{x_1}+\int_{x_2}^{x_4}\right)f(\xi)f'(\xi)[1+f'^2(\xi)]^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}4(4r\rho)^{-\frac{1}{2}}(\alpha^2+\sin^2\lambda)^{-\frac{1}{2}}d\lambda d\xi.$$

To simplify this integral we note:

(A) Replacing $[1+f'^2(\xi)]^{-\frac{1}{2}}$ by 1 introduces an error of

$$< \text{const.}\, x^{3/5}(f(x)/x)^3(f(x)/\delta_n) = O(x^{-\frac{1}{2}}),$$

where (1), (2), (3) and (1.4) have been used.

(B) Replacing $\rho^{\frac{1}{2}}$ by $r^{\frac{1}{2}}$ introduces an error of

$$< \text{const.}\, x^{3/5}[f^{3/2}(x)/x][f(x)/\delta_n]\,\text{Max}\,|\,r-\rho\,|/f(x)\,(r^{\frac{1}{2}}+\rho^{\frac{1}{2}})$$

$$< \text{const.}\, [f(x)/x^{2/5}\delta_n][x^{3/5}f(x)/x],$$

by the mean-value theorem, $= O(x^{-1/5})$.

(C) Replacing α^2 by $(x-\xi)^2/4r\rho$ introduces an error of

$$< (kf(x)/x)\left(\int_{x_3}^{x_1}+\int_{x_2}^{x_4}\right)\int_0^{\frac{1}{2}\pi}[(x-\xi)^2/x^2][(x-\xi)^2/4r\rho+\sin^2\lambda]^{-3/2}d\lambda d\xi$$

$$= O[(f(x)/x)x^{3/5}x^{-2}f^3(x)/\delta_n] = O(x^{-\frac{1}{2}}).$$

(D) Replacing ρ by r in this same radical gives an error of

$$< [kf(x)/x]\left(\int_{x_3}^{x_1}+\int_{x_2}^{x_4}\right)\int_0^{\frac{1}{2}\pi}[(x-\xi)^2/4r]\,|\,1/\rho-1/r\,|$$

$$\times[(x-\xi)^2/4r\rho+\sin^2\lambda]^{-\frac{1}{2}}[(x-\xi)^2/4r^2+\sin^2\lambda]^{-\frac{1}{2}}$$

$$\times\{[(x-\xi)^2/4r\rho+\sin^2\lambda]^{\frac{1}{2}}+[(x-\xi)^2/4r^2+\sin^2\lambda]^{\frac{1}{2}}\}^{-1}d\lambda d\xi$$

$$< [kf(x)/x]\left(\int_{x_3}^{x_1}+\int_{x_2}^{x_4}\right)\int_0^{\frac{1}{2}\pi}[(x-\xi)^2/4r]x^{3/5}[1/x\,f(x)](\tfrac{1}{2})$$

$$\times[(x-\xi)^2/8r^2+\sin^2\lambda]^{-3/2}d\lambda d\xi$$

$= O(x^{-1/5})$, where k, as above, denotes a constant and where the mean-value theorem has again been used.

(E) Replacing $f(\xi)f'(\xi)$ by $f(x)f'(x)$ gives an error

$$< \left(\int_{x_3}^{x_1}+\int_{x_2}^{x_4}\right)[\,|\,f(\xi)f'(\xi)-f(x)f'(x)\,|/f(x)]$$

$$\int_0^{\frac{1}{2}\pi}2f(x)[(x-\xi)^2+4r^2\sin^2\lambda]^{-\frac{1}{2}}\,d\lambda d\xi.$$

Now for $\xi < x$, $|\,f(\xi)f'(\xi)-f(x)f'(x)\,| < f(x_3)f'(x_3)-f(x)f'(x)$

$$= [f^2(x_3)/2x_3 - f^2(x)/2x] - [f^2(x_3)/(4x_3 \log x_3) - f^2(x)/(4x \log x)]$$
$$+ \{[f^2(x_3) \log\log x_3]/(8x_3 \log^2 x_3) - [f^2(x) \log\log x]/(8x \log^2 x)\}$$
$$+ O(\log^{-5/2} x)$$

$= O(x^{-2/5}) + O(\log^{-5/2} x)$, where the mean value theorem is used to estimate the three differences, and (1) and (1.4) are used as usual. An almost identical calculation obtains the same result for $\xi > x$; accordingly the above error is

$$< \text{const.}\ (\log^{-5/2} x) \int_0^{\frac{1}{2}\pi} \left(\int_{x_3}^{x_1} + \int_{x_2}^{x_4} \right) [(x-\xi)^2 + 4r^2 \sin^2 \lambda]^{-\frac{1}{2}} d\xi d\lambda$$

$$= \text{const.}\ (\log^{-5/2} x) \int_0^{\frac{1}{2}\pi} \log\{[x^{3/5} + (x^{6/5} + 4r^2 \sin^2 \lambda)^{\frac{1}{2}}]/[\delta_n + (\delta^2{}_n + 4r^2 \sin^2 \lambda)^{\frac{1}{2}}]\} d\lambda$$

$$< \text{const.}\ (\log^{-5/2} x) \int_0^{\frac{1}{2}\pi} \log x \, d\lambda = O(\log^{-3/2} x).$$

At this point then we have:

$$R_2 = (-2/\pi) \left(\int_{x_3}^{x_1} + \int_{x_2}^{x_4} \right) f(x) f'(x) \int_0^{\frac{1}{2}\pi} [(x-\xi)^2 + 4r^2 \sin^2 \lambda]^{-\frac{1}{2}} d\lambda d\xi + O(\log^{-3/2} x)$$

$$= (-2/\pi) f(x) f'(x) \int_0^{\frac{1}{2}\pi} 2 \int_{\delta_n}^{x^{3/5}} (p^2 + 4r^2 \sin^2 \lambda)^{-\frac{1}{2}} dp d\lambda + O(\log^{-3/2} x)$$

after changing the order of integration and making a simple change of variable. Now integrating,

$$R_2 = (-4/\pi) f(x) f'(x) \int_0^{\frac{1}{2}\pi} \log\{[x^{3/5} + (x^{6/5} + 4r^2 \sin^2 \lambda)^{\frac{1}{2}}]/[\delta_n + (\delta^2{}_n + 4r^2 \sin^2 \lambda)^{\frac{1}{2}}]\} d\lambda + O(\log^{-3/2} x).$$

Since

$$\log[x^{3/5} + (x^{6/5} + 4r^2 \sin^2 \lambda)^{\frac{1}{2}}] = (3/5) \log x + \log 2 + O(x^{-1/5})$$

we find

$$R_2 = -2f(x) f'(x) [(3/5) \log x + \log 2]$$
$$+ (4/\pi) f(x) f'(x) \int_0^{\frac{1}{2}\pi} \log [\delta_n + (\delta^2{}_n + 4r^2 \sin^2 \lambda)^{\frac{1}{2}}] d\lambda + O(\log^{-3/2} x)$$
$$= -2f(x) f'(x) [(3/5) \log x + \log 2] + O(\log^{-3/2} x)$$
$$+ (4/\pi) f(x) f'(x) [\tfrac{1}{2}\pi \log 2f(x) + \int_0^{\frac{1}{2}\pi} \log(a + (a^2 + \sin^2 \lambda)^{\frac{1}{2}}) d\lambda],$$

where $a = \delta_n/2r = O(\log^{\frac{1}{2}-n} x)$ and so is small.

Consider $\int_0^{\frac{1}{2}\pi} [\log(a + (a^2 + \sin^2 \lambda)^{\frac{1}{2}}) - \log \sin \lambda] d\lambda$. This tends to zero with a. However, a calculation involving the Maclaurin series in powers of a leads to a divergent integral; so we let $a = b^2$ and have by the mean value theorem:

$$| \log [b^2 + (b^4 + \sin^2 \lambda)^{\frac{1}{2}}] - \log \sin \lambda \,| < |\, b \,| \operatorname{Max} |\, 2b/(b^4 + \sin^2 \lambda)^{\frac{1}{2}} \,|$$

$$= |\, b \,| (2/\sin \lambda)^{\frac{1}{2}} = (2a/\sin \lambda)^{\frac{1}{2}},$$

so that

$$\int_0^{\frac{1}{2}\pi} [\log(a + (a^2 + \sin^2 \lambda)^{\frac{1}{2}}) - \log \sin \lambda] d\lambda < \int_0^{\frac{1}{2}\pi} (2a/\sin \lambda)^{\frac{1}{2}} d\lambda$$

$$= O(a^{\frac{1}{2}}) = O(\log^{\frac{1}{4}-n/2} x),$$

or

$$\int_0^{\frac{1}{2}\pi} \log(a + (a^2 + \sin^2 \lambda)^{\frac{1}{2}}) d\lambda = \int_0^{\frac{1}{2}\pi} \log \sin \lambda d\lambda + O(\log^{\frac{1}{4}-n/2} x)$$

$$= (-\pi/2) \log 2 + O(\log^{\frac{1}{4}-n/2} x).$$

Using this in R_2 gives

$$R_2 = -2f(x)f'(x)[(3/5)\log x + \log 2] + (4/\pi)f(x)f'(x)[(\pi/2)\log(2r) - (\pi/2)\log 2] + O(\log^{-\frac{1}{4}-n/2} x) + O(\log^{-3/2} x),$$

or

$$R_2 = f(x)f'(x)[\log f^2(x) - (6/5)\log x - \log 4] + O(\log^{-\frac{1}{4}-n/2} x) + O(\log^{-3/2} x).$$

And now

$$R_3 = -\frac{1}{2\pi}\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right) f(\xi)f'(\xi)[1 + f'^2(\xi)]^{-\frac{1}{2}} \times \int_0^{\frac{1}{2}\pi} 4[(x-\xi)^2 + (r+\rho)^2 - 4r\rho\cos^2\lambda]^{-\frac{1}{2}} d\lambda d\xi.$$

In this last part of I_2 we may use the series for the elliptic integral so that, since $|\, x - \xi \,| \geq x^{3/5}$,

$$\int_0^{\frac{1}{2}\pi} [(x-\xi)^2 + (r+\rho)^2 - 4r\rho\cos^2\lambda]^{-\frac{1}{2}} 4 d\lambda$$

$$= 2\pi[(x-\xi)^2 + (r+\rho)^2]^{-\frac{1}{2}}[O(f^2(x)/x^{6/5}) + 1].$$

This gives

$$R_3 = -\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right) f(\xi) f'(\xi)[1 + f'^2(\xi)]^{-\frac{1}{2}}[(x-\xi)^2 + (r+\rho)^2]^{-\frac{1}{2}} \times [1 + O(x^{-1/5})]d\xi.$$

The error term contributes

$$< \text{const.}\ [f^2(x)/x][f^2(x)/x^{6/5}] \int_{x_4}^{5x/4} d\xi/|\,x-\xi\,| = O(x^{-1/5}).$$

To simplify the other integrals note:

(A) Replacing $[1 + f'^2(\xi)]^{-\frac{1}{2}}$ by 1 gives an error of

$$< \text{const.}\ x[f^2(x)/x^2]x^{-3/5} = O(x^{-3/5}).$$

(B) Replacing $[(x-\xi)^2 + (r+\xi)^2]^{-\frac{1}{2}}$ by $|\,x-\xi\,|^{-1}$ gives an error of

$$< \text{const.}\ [f^2(x)/x][f^2(x)/x^{6/5}] \int_{x_4}^{5x/4} d\xi/|\,x-\xi\,| = O(x^{-1/5}).$$

With these changes

$$R_3 = -\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right) f(\xi) f'(\xi)\,|\,x-\xi\,|^{-1} d\xi + O(x^{-1/5}).$$

Using (1), this becomes

$$R_3 = -\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right)[f^2(\xi)/2\xi][1 - (1/(2\log\xi)) + (\log\log\xi)/(4\log^2\xi)]d\xi/|\,\xi - x\,| + O(\log^{-3/2} x).$$

Now for $\xi > x$,

$$|\,f^2(x)/x - f^2(\xi)/\xi\,| \leq f^2(x)/x - f^2(5x/4)/(5x/4) = O(\log^{-3/2} x),$$

where we use (1.4) exactly as before; for $\xi < x$, the argument is again almost the same and the result identical. If then in R_3 we replace $f^2(\xi)/\xi$ by $f^2(x)/x$, the error introduced into the first term is not small enough, but in the last two terms it is

$$< \text{const.}\ \log^{-5/2} x \int_{x_4}^{5x/4} d\xi/(\xi - x) = O(\log^{-3/2} x).$$

Again using identities very similar to A, B, and C we find almost at once that replacing $1/\log\xi$ by $1/\log x$ and $\log\log\xi/\log^2\xi$ by $\log\log x/\log^2 x$ introduces errors which are $O(\log^{-3/2} x)$. Accordingly,

$$R_3 = -\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right)\{[f^2(\xi)/2\xi] + [f^2(x)/2x][(-1/(2\log x)) + (\log\log x)/(4\log^2 x)]\}(d\xi/|\,\xi - x\,| + O(\log^{-3/2} x)$$

$$= -\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right) (f^2(\xi)/2\xi)\,d\xi/|\,\xi - x\,|$$

$$- [f^2(x)/2x][-2/5 + (\log\log x)/(5\log x)] + O(\log^{-3/2} x).$$

Calling the remaining integrals R_4,

$$R_4 = -\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right)[f^2(x)/2x]\,d\xi/|\,\xi - x\,|$$

$$+\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right)[f^2(x)/2x - f^2(\xi)/2\xi]\,d\xi/|\,\xi - x\,|$$

$$= -[f^2(x)/2x][(4/5)\log x - 2\log 4]$$

$$-\left(\int_{3x/4}^{x_3} + \int_{x_4}^{5x/4}\right)\int_x^\xi [(f(t)f'(t)/t) - (f^2(t)/2t^2)]\,dt\,d\xi/|\,\xi - x\,|$$

$$= -[f^2(x)/2x][(4/5)\log x - 2\log 4]$$

$$-\left(\int_x^{x_4}\int_{x_4}^{5x/4} + \int_{x_4}^{5x/4}\int_t^{5x/4} + \int_x^{x_3}\int_{3x/4}^{x_3} + \int_{x_3}^{3x/4}\int_{3x/4}^{t}\right)$$

$$[(f(t)f'(t)/t) - (f^2(t)/2t^2)]\,d\xi\,dt/|\,\xi - x\,|.$$

Now by (1),

$$(f(t)f'(t)/t) - (f^2(t)/2t^2)$$

$$= [f^2(t)/2t^2][(-1/(2\log t)) + (\log\log t)/(4\log^2 t) + O(\log^{-2} t)].$$

Since the first and third integrals have constant limits they can be calculated directly and are $O(\log^{-3/2} x)$, so that

$$R_4 = -[f^2(x)/2x][(4/5)\log x - 2\log 4]$$

$$-\left(\int_{x_4}^{5x/4}\int_t^{5x/4} + \int_{x_3}^{3x/4}\int_{3x/4}^{t}\right)[f^2(t)/2t^2][(-1/(2\log t))$$

$$+ (\log\log t)/(4\log^2 t)]\,d\xi\,dt/|\,\xi - x\,| + O(\log^{-3/2} x)$$

$$= -[f^2(x)/2x][(4/5)\log x - 2\log 4]$$

$$-\left(\int_{x_4}^{5x/4} + \int_{x_3}^{3x/4}\right)[f^2(t)/2t^2][(-1/(2\log t))$$

$$+ (\log\log t)/(4\log^2 t)]\log[(x/4)/|\,x - t\,|]\,dt + O(\log^{-3/2} x).$$

Again as before, for $t > x$,

$$|(f^2(x)/x\log x) - (f^2(t)/t\log t)|$$
$$< [f^2(x)/x\log x] - [f^2(5x/4)/(5x/4)\log(5x/4)]$$
$$= O(\log^{-5/2} x);$$

similarly for $t < x$. Also

$$|[f^2(x)\log\log x/x\log^2 x] - [f^2(t)(\log\log t)/(t\log^2 t)]| = O(\log^{-5/2} x)$$

both for $t > x$ and $t < x$. In R_4 these differences contribute $O(\log^{-3/2} x)$ so that

$$R_4 = -[f^2(x)/2x][(4/5)\log x - 2\log 4]$$
$$+ \{[f^2(x)/(4x\log x)] - [f^2(x)(\log\log x)/(8x\log^2 x)]\}$$
$$\left(\int_{x_4}^{5x/4} + \int_{x_3}^{3x/4}\right)\log[(x/4)/|x-t|]t^{-1}dt + O(\log^{-3/2} x)$$
$$= -[f^2(x)/2x][(4/5)\log x - 2\log 4] + O(\log^{-3/2} x),$$

since direct evaluation of the remaining integrals shows them to be $O(1)$.

In summary then

$$I_2 = \int_{3x/4}^{x} f(\xi)f'(\xi)/(\xi+1)\,d\xi$$
$$+ [f^2(x)/2x][\log(5/4) + 2/5 - (\log\log x)/(5\log x)$$
$$- (4/5)\log x + 2\log 4]$$
$$+ f(x)f'(x)[\log f^2(x) - (6/5)\log x - \log 4] + O(\log^{-3/2} x)$$
$$= \int_{3x/4}^{x} f(\xi)f'(\xi)/(\xi+1)\,d\xi$$
$$+ [f^2(x)/2x][\log(f^2(x)/x^2) + \log 5 + \tfrac{1}{2}] + O(\log^{-3/2} x).$$

Next we consider

$$I_3 = -\tfrac{1}{2\pi}\iint_{W_3}(1/R - 1/1+\xi)\phi_n dS$$

$$1/R = [(x-\xi)^2 + r^2 + \rho^2 - 2r\rho\cos(\theta-\lambda)]^{-\frac{1}{2}}$$
$$= (1/\xi - x)[1 + O(f^2(\xi)/\xi^2)].$$
$$= 1/(\xi - x) + O(1/\xi^2).$$

So

$$I_3 = -\tfrac{1}{2\pi}\int\int_{W_3}[(x+1)/(\xi-x)(\xi+1)+O(1/\xi^2)][f'(\xi)+O(f'^3(\xi))]dS$$

$$= -\int_{5x/4}^{\infty}[(x+1)/(\xi-x)(\xi+1)]f(\xi)f'(\xi)d\xi + O(\int_{5x/4}^{\infty}(f^2(\xi)/\xi)d\xi/\xi^2)$$

$$= -\int_{5x/4}^{\infty}[(x+1)/(\xi-x)(\xi+1)]f(\xi)f'(\xi)d\xi + O(1/x).$$

Replacing $x+1$ by x and $\xi+1$ by ξ both give errors less than

$\int_{5x/4}^{\infty} d\xi/\xi^2 = O(1/x)$, so that using (1),

$$I_3 = -\int_{5x/4}^{\infty}[x/\xi(\xi-x)]f(\xi)f'(\xi)d\xi + O(1/x)$$

$$= -\int_{5x/4}^{\infty}[x/\xi(\xi-x)][f^2(\xi)/2\xi]d\xi + O(\log^{-3/2} x),$$

since

$$x\int_{5x/4}^{\infty}1/\xi(\xi-x)d\xi < x\int_{5x/4}^{\infty}5/\xi^2\,d\xi = 4.$$

Now

$$\int_{5x/4}^{\infty}f^2(\xi)\,d\xi/2\xi^2(\xi-x) = \int_{5x/4}^{\infty}f^2(x)\,d\xi/2x\xi(\xi-x)$$

$$+\int_{5x/4}^{\infty}\int_{x}^{\xi}[f(t)f'(t)/t - f^2(t)/2t^2]dt\,d\xi/\xi(\xi-x)$$

$$= [f^2(x)/2x][(\log 5)/x]$$

$$\left(\int_{x}^{5x/4}\int_{5x/4}^{\infty}+\int_{5x/4}^{\infty}\int_{t}^{\infty}\right)O(1)/(2t\log^{3/2} t)\,d\xi dt/\xi(\xi-x),$$

where we have used (1) and (1. 4) and changed the order of integration. Of the two remaining integrals the first is

$$< \tfrac{1}{2}\log^{-3/2} x\int_{x}^{\infty}\int_{5x/4}^{\infty}O(1)\,(5d\xi/\xi^2)\,dt/t = O(1/x\log^{-3/2} x),$$

and the second is

$$< \tfrac{1}{2}\log^{-3/2} x\int_{5x/4}^{\infty}\int_{t}^{\infty}O(1)\,(5d\xi/\xi^2)\,dt/t = O(1/x\log^{-3/2} x)\,;$$

and so $I_3 = -\log 5\, f^2(x)/2x + O(\log^{-3/2} x)$.

Finally we treat $J_2 = \frac{1}{2\pi} \int\int_{W_2} \phi(\xi, \rho) \cos \psi / R^2 dS$. Using (6. 5),

$$J_2 = \tfrac{1}{2\pi} [C_2 + \tfrac{1}{2} B(x)] \int\int_{W_2} \cos \psi / R^2 \, dS$$

$$- (1/4\pi) \int\int_{W_2} [B(x) - B(\xi)] \cos \psi / R^2 \, dS + O(x^{-\frac{1}{2}}),$$

and now by (3. 15)

$$J_2 = - C_2 - \tfrac{1}{2} B(x) + O(B(x)/x) + S_1 + O(x^{-\frac{1}{2}}),$$

where S_1 is the remaining integral. By (1) and (1. 4)

$$B(x) = \int_a^x f'^2(t) \, dt < \text{const.} \int_a^x 1/(t \log^{\frac{1}{2}} t) \, dt = O(\log x),$$

so that the first error term in J_2 above is $O(x^{-\frac{1}{2}})$.

Now let W_{21} be that portion of W_2 for which ξ is between $x - b$ and $x + b$, and let W_{22} be the remaining two portions. Then in W_{21}

$$| B(x) - B(\xi) | = | \int_\xi^x f'^2(t) \, dt | \leq | \int_\xi^x c^2/(t \log^{\frac{1}{2}} t) \, dt |$$

$= O[b/(x \log^{\frac{1}{2}} x)] = O(\log^{-n} x)$, so that from this region S_1 receives a contribution of

$$< k \log^{-n} x \int\int_{W_{21}} - \cos \psi / R^2 \, dS < k \log^{-n} x \int\int_{W_2} - \cos \psi / R^2 \, dS$$

$$= O(\log^{-n} x),$$

k being a constant and n arbitrary.

Furthermore, since $B(x) = O(\log x)$, the contribution to S_1 from the region W_{22} is

$$< k \log x \int\int_{W_{22}} f(\xi) b^{-2} d\xi d\lambda = O(x \log^{2n+1} x / x^{3/2}) = O(x^{-1/3}).$$

Consequently $J_2 = - C_2 - \frac{1}{2} B(x) + O(\log^{-n} x)$.

Part II. Calculation of New Terms.

Inserting (6. 5), (6. 6), (6. 7) and our reappraisals of I_1, I_2, I_3, and J_2 into (3. 10) we have

$$B(x) = \int_a^x f(\xi) f'(\xi) / (\xi + 1) \, d\xi + \text{const.} + [f^2(x)/x] \log [f(x)/x]$$

$$+ \tfrac{1}{2} dc^2 [\log^{-\frac{1}{2}} x - (\log \log x)/(4 \log^{3/2} x)] + O(\log^{-3/2} x),$$

where $d = \frac{1}{2} - \log 4$; now by using (6.15) and (6.17) this becomes

$$E_1 - F_1 = (1/4) \int_a^x f^2(\xi)/\xi^2 \, d\xi - [f^2(x)/x] \log [x/f(x)] - K/4$$
$$+ \tfrac{1}{2} dc^2 [\log^{-\frac{1}{2}} x - (\log \log x)/(4 \log^{3/2} x)] + O(\log^{-3/2} x),$$

where K is a constant. By (6.15) and (6.17)

$$E_1 - F_1 = \int_x^\infty [f(\xi) f'(\xi)/(\xi + 1) - f^2(\xi)/4\xi^2 - f'^2(\xi)] d\xi.$$

As usual, replacing $\xi + 1$ by ξ gives an error of $O(1/x)$, so using (1)

$$E_1 - F_1 = \int_x^\infty O(f^2(\xi)/\xi^2 \log^2 \xi) d\xi + O(1/x) = O(\log^{-3/2} x).$$

Accordingly, letting $f(x) = x^{\frac{1}{2}} h(x)/\log^{1/4} x$, we have $\int_a^x h^2(\xi)/\xi \log^{\frac{1}{2}} \xi \, d\xi$

$$= 4h^2(x) \log^{-\frac{1}{2}} x [\tfrac{1}{2} \log x + (1/4) \log \log x - \log h(x)] + K$$
$$- 2dc^2 [\log^{-\frac{1}{2}} x - (\log \log x)/(4 \log^{3/2} x)] + O(\log^{-3/2} x).$$

Substituting $y = \log x$, $h^2(x) = H(y)$, $b = \log a$, and using (6.19), this becomes

$$R(y) = K + y^{-\frac{1}{2}} (\log y) H(y) - 2y^{-\frac{1}{2}} H(y) \log H(y)$$
$$- 2dc^2 [y^{-\frac{1}{2}} - y^{-3/2} (\log y)/4] + O(y^{-3/2}),$$

and so by (1.4)

$$R(y) = K + O(y^{-3/2+\epsilon}) + c^2 y^{-\frac{1}{2}} \log y - (4c^2 \log c + 2dc^2) y^{-\frac{1}{2}}.$$

Using this expression for $R(y)$ in (6.20), carrying out the integrations and combining terms, we find:

$$H(y) = c^2 [1 - \log y/4y + (\log c + 1/4 + d/2)/y] + O(y^{-2+\epsilon}).$$

Returning to our first expression for $R(y)$ above, and using this latest value for $H(y)$, direct substitution followed by recombination of terms gives:

$$R(y) = K + c^2 y^{-\frac{1}{2}} \log y - 2c^2 (2 \log c + d) y^{-\frac{1}{2}} - c^2 y^{-3/2} (\log^2 y)/4$$
$$+ c^2 (2 \log c + 3/4 + d) y^{-3/2} \log y + O(y^{-3/2}).$$

Now substituting this value of $R(y)$ into (6.20), carrying out the integrations and combining terms, we find

$$H(y) = c^2 [1 - \log y/4y + (\log c + 1/4 + d/2)/y]$$
$$+ 3 \log^2 y/32y^2 - [3(\log c)/4 + 5/16 + 3d/8] \log y/y^2 + O(1/y^2).$$

Using the binomial theorem and recombining,

$$(H(y))^{\frac{1}{2}} = h(x) = c[1 - \log y/8y + ((\log c)/2 + 1/8 + d/4)/y$$
$$+ 5\log^2 y/128y^2 - (5(\log c)/16 + 9/64 + 5d/32)\log y/y^2 + O(1/y^2)].$$

Finally, recalling that $y = \log x$ and that $f(x) = x^{\frac{1}{2}}h(x)/\log^{1/4} x$ we have for $f(x)$ the expression written at length in the introduction.

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[1] Norman Levinson, "On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid," *Annals of Mathematics*, vol. 47 (1946), pp. 704-730.

[2] Decimal equation numbers are as in the paper of Levinson.

ON ORDINARY DIFFERENTIAL EQUATIONS OF ANY EVEN ORDER AND THE CORRESPONDING EIGENFUNCTION EXPANSIONS.*

By Kunihiko Kodaira.

The general theory of expanding an arbitrary function in terms of the eigenfunctions of a singular differential operator of the second order was first given by H. Weyl.[1] An alternative method, based on the general theory of linear transformations in the Hilbert space, is to be found in a treatise by M. H. Stone.[2] Recently E. C. Titchmarsh has treated the same theory by still another method and obtained some new results of importance for applications.[3] The purpose of the present paper is to generalize the Weyl-Stone-Titchmarsh theory of eigenfunction expansions to the case of formally self-adjoint ordinary differential operators with real coefficients of any even order. Our method is also based on the general theory of linear transformations in the Hilbert space and goes along the lines of H. Weyl.[4] In Section **1** we shall give preliminary remarks on the formally self-adjoint differential operator of real coefficients, which will be denoted by L. In Section **2** we shall analyse the singularities of the operator L. For that purpose, we introduce a complex projective space $\mathfrak{P}$, each point of which is associated with a general solution of the differential equation $L[u] = l \cdot u$, and construct the subsets $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$ of $\mathfrak{P}$ corresponding to the "limit circles" in the theory of H. Weyl.[5] Furthermore it will be shown that the formally self-adjoint differential operators of the even order n can be classified into $(\frac{1}{2}n+1)^2$ different types according to the nature of their singularities. In Section **3** we shall introduce Green's function and determine the closure and the adjoint of the operator L. The boundary conditions will be discussed in Section **4**. It will be shown that, under suitable boundary conditions, L becomes a self-adjoint operator. The next section, **5**, will be devoted to the proof of the spectral theorem, which can be regarded as a generalization of the Weyl-Stone-Titchmarsh results. In

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* Received September 22, 1949.

[1] Weyl, [10], [11], [12]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Stone, [7], Chap. X, §3.

[3] Titchmarsh, [9]. Another proof of Titchmarsh's results is given in Kodaira [3].

[4] Weyl, [11], [12].

[5] Weyl, [11], pp. 221-231.

Section **6**, the expansion theorem will be deduced from the spectral theorem. Finally it will be shown in Section **7** that the spectral and the expansion theorems can be readily extended to the case of simultaneous differential equations.

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1. Introduction. Consider a formal differential operator

$$L = p_0(x)\,(d/dx)^n + p_1(x)\,(d/dx)^{n-1} + \cdots + p_n(x), \qquad (a < x < b)$$

of the even order $n = 2\nu$ defined in a (finite or infinite) *open* interval (a, b), where each coefficient $p_m(x)$ $(m = 0, 1, \cdots, n)$ is a *real valued continuous function defined in* (a, b) *having continuous derivatives up to the order* $n - m$ *and* $p_0(x) > 0$ [or $p_0(x) < 0$]; for $x \to a$ or $x \to b$, $p_m(x)$ may behave arbitrarily, e. g. increase infinitely, oscillate infinitely many times.[6] Furthermore we assume that L is *formally self-adjoint*, i. e. L coincides with its Lagrange adjoint or

$$(1.1)_m \quad p_m(x) = \sum_{k=0}^{m-1} (-1)^k C_{m-k}{}^{n-k} p_k{}^{(m-k)}(x) + (-1)^m p_m(x), \qquad (m = 1, 2, \cdots, n),$$

where $C_r{}^s$ mean binomial coefficients. For *odd* m, $(1.1)_m$ is rewritten as

$$(1.2)_m \quad 2p_m(x) = C_m{}^n p_0{}^{(m)}(x) - C_{m-1}{}^{n-1} p_1{}^{(m-1)}(x) + - \cdots + C_1{}^{n-m+1} p'_{m-1}(x), \qquad (m = 1, 3, 5, \cdots, n-1),$$

while, *for even* m, $(1.1)_m$ *follows from* $(1.1)_1, (1.1)_2, \cdots, (1.1)_{m-1}$. Hence, in general, L *is formally self adjoint if the coefficients* $p_0(x), \cdots, p_{n-1}(x)$ *satisfy the conditions* $(1.2)_m$, $m = 1, 3, \cdots, n-1$. Thus, in our operator L, *the coefficients* $p_0(x), p_2(x), p_4(x), \cdots, p_n(x)$ *may be chosen arbitrarily, while* $p_1(x), p_3(x), \cdots$ *must be determined successively by* $(1.2)_1, (1.2)_3, \cdots, (1.2)_{n-1}$.

Put now

$$L[u] = p_0(x)\,(d^n u/dx^n) + p_1(x)\,(d^{n-1}u/dx^{n-1}) + \cdots + p_n(x) \cdot u$$

for arbitrary functions $u(x)$ having continuous derivatives up to the order n. Then we have *Green's formula*

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[6] These conditions on the coefficients may be replaced by weaker ones, but in the present paper we are not interested with such generalizations. Cf. Stone, [7], pp. 448-453, Halperin, [2].

$$(1.3) \qquad \int_y^x (L[u]\cdot v - u\cdot L[v])\,dx = [uv](x) - [uv](y),$$

where $[uv](x)$ means the *bilinear form*

$$[uv](x) = \sum_{j+k\leqq n-1} B_{jk}(x)\,u^{(j)}(x)\,v^{(k)}(x)$$

of two *"vectors"* $(u, u', u'', \cdots, u^{(n-1)})$, $(v, v', v'', \cdots, v^{(n-1)})$ with the coefficients

$$(1.4) \qquad B_{jk}(x) = (-1)^{j+1} \sum_{h=0}^{n-j-k-1} (-1)^h C_k^{n-j-1-h}\, p_h{}^{(n-j-1-k-h)}.$$

The bilinear form $[uv](x)$ is obviously skew-symmetric, i. e.

$$(1.5) \qquad [uv](x) = -[vu](x).\text{[7]}$$

For $j + k = n - 1$, we have

$$(1.6) \qquad B_{jk}(x) = (-1)^{j+1} p_0(x), \qquad (j+k=n-1).$$

The bilinear form $[uv](x)$ is therefore non-degenerate, since by hypothesis $p_0(x) \neq 0$. Now we introduce the *"Wronskian"*

$$[u_1u_2\cdots u_n](x) = \{-p_0(x)\}^\nu \cdot \det(u_j{}^{(k-1)}), \qquad (j, k = 1, 2, \cdots, n).$$

Then we have the identity

$$(1.7) \qquad (1/2^\nu \nu!)\Sigma \pm [u_1u_{\nu+1}][u_2u_{\nu+2}]\cdots[u_\nu u_n] = [u_1u_2\cdots u_n],$$

where $\Sigma \pm$ means the alternating sum extending to all $n!$ permutations of n functions $u_1, u_2, \cdots, u_n$. (1.7) is proved as follows:[8] As one readily infers, the left hand side of (1.7) is equal to

$$[u_1u_2\cdots u_n]\cdot(-1)^{\nu(\nu+1)/2} p_0{}^{-\nu} (1/2^\nu \nu!)\Sigma \pm B_{01}B_{23}\cdots B_{n-2\ n-1},$$

$\Sigma \pm$ being the alternating sum extending to all $n!$ permutations of the suffixes $0, 1, 2, \cdots, n-1$; while, since $B_{jk} = 0$ for $j + k \geqq n$, we get, using (1.6),

$$(1/2^\nu \nu!)\Sigma \pm B_{01}B_{23}\cdots B_{n-2\ n-1} = B_{0\ n-1}B_{1\ n-2}\cdots B_{\nu-1\ \nu} = (-1)^{\nu(\nu+1)/2} p_0{}^\nu.$$

Hence we have (1.7).

2. Classification.

We shall consider the *differential equation*

$$(2.1) \qquad L[u] = l\cdot u,$$

where l means a complex parameter. From (1.3) follows that, *if u and v*

[7] This identity can be deduced readily from Green's formula.
[8] Weyl, [14], pp. 166-167.

satisfy one and the same equation (2.1), $[uv](x)$ *does not depend on* x. Combined with (1.7), this shows further that *the Wronskian* $[u_1u_2\cdots u_n](x)$ *of* n *solutions* $u_1(x), u_2(x), \cdots, u_n(x)$ of the *equation* (2.1) *is a constant.* In these cases, we write therefore $[uv]$ or $[u_1u_2\cdots u_n]$ for $[uv](x)$ or $[u_1\cdots u_n](x)$, respectively.

DEFINITION 2.1. *By a regular system of fundamental solutions of the differential equation* $L[u]=l\cdot u$ *we shall mean a system of* n *solutions* $s_j(x,l)$ $(j=1,2,\cdots,n)$ *such that*

(2.2) $$[s_1s_2\cdots s_n]=1,$$ (2.3) $$s_j(x,\bar{l})=\bar{s}_j(x,l),$$

and that the functions $d^ms_j(x,l)/dx^m$ $(j=1,2,\cdots,n;\ m=0,1,\cdots,n-1)$ *are entire functions of* l. *A regular system of fundamental solutions will be called a canonical system, if the skew symmetric matrix* $([s_js_k])$ *has the "canonical form" in the sense that*

(2.4) $$[s_js_k]=\epsilon_{jk},$$

ϵ_{jk} *being the quantities defined by* $\epsilon_{jk}=1$ (*for* $k=j+\nu$), $=-1$ (*for* $k=j-\nu$), $=0$ (*otherwise*).

It should be noted here that (2.4) *implies* (2.2), as (1.7) shows.

A canonical system of fundamental solutions is obtained, for example, as follows: Choose a fixed point c, $a<c<b$, arbitrarily (in what follows c means always a fixed point such that $a<c<b$). Then there exist n *real* vectors

$$e_j=(e_j, e'_j, e''_j, \cdots, e_j^{(n-1)}), \qquad (j=1,2,\cdots,n)$$

such that $[e_je_k](c)=\epsilon_{jk}$. Now, under the boundary conditions

$$[d^mu/dx^m]_{x=c}=e_j^{(m)}, \qquad (m=0,1,\cdots,n-1),$$

the differential equation $L[u]=l\cdot u$ has one and only one solution, which will be denoted by $s_j^0(x,l)$. As one readily infers, the functions $s_j^0(x,l)$ $(j=1,2,\cdots,n)$ thus defined constitute a canonical system.

Obviously an arbitrary regular system of fundamental solutions $s_1, s_2, \cdots, s_n$ is obtained from the canonical system defined above by a unimodular transformation

(2.5) $$s_j(x,l)=\sum_{k=1}^{n}U_j^k(l)s_k^0(x,l), \qquad (\det(U_j^k)=1),$$

where $U_j^k(l)$ $(j,k=1,\cdots,n)$ are entire functions of l such that $U_j^k(\bar{l})=\bar{U}_j^k(l)$.

Suppose now a regular system of fundamental solutions $s_j(x,l)$ $(j=1,$

$2, \cdots, n)$ as given. Then every solution $w(x)$ of the differential equation $L[w] = l \cdot w$ is represented as

$$w(x) = f^1 s_1(x,l) + f^2 s_2(x,l) + \cdots + f^n s_n(x,l).$$

We associate with such $w(x)$ the vector $f = (f^1, \cdots, f^n)$ and denote $w(x)$ by $w(x,l,f)$. For the given solution $w(x) = w(x,l,f)$, the vector $f = (f^1, \cdots, f^n)$ depends clearly on the choice of the fundamental solutions $s_1, s_2, \cdots, s_n$: *under the unimodular transformation*

$$\text{(U)} \qquad \tilde{s}_j(x,l) = \Sigma\, U_j{}^k(l) s_k(x,l), \qquad (\det(U_j{}^k) = 1)$$

$U_j{}^k(l)$ being entire functions of l such that $U_j{}^k(\bar{l}) = \bar{U}_j{}^k(l)$, f^j are transformed contragrediently to s_j, i. e.

$$\text{(V)} \qquad \tilde{f}^j = \Sigma\, V_k{}^j(l) f^k,$$

where $(V_k{}^j(l))$ means the inverse matrix of $(U_j{}^k(l))$. Obviously $V_k{}^j(l)$ *are also entire functions of l and satisfy $V_k{}^j(\bar{l}) = \bar{V}_k{}^j(l)$.*

Now we introduce the $(n-1)$*-dimensional projective space* $\mathfrak{P}$ *and consider $(f^1, f^2, \cdots, f^n)$ (excluding $(0, 0, \cdots, 0)$) as the homogeneous coordinates of a point in $\mathfrak{P}$ which will be denoted by (f).* If there is no possibility of confusion, *we write simply f for (f)*; especially, for an arbitrary subset $\mathfrak{S}$ of $\mathfrak{P}$, *we write $f \,\varepsilon\, \mathfrak{S}$ for $(f) \,\varepsilon\, \mathfrak{S}$.*

Choose a fixed point c, $a < c < b$, arbitrarily and, for arbitrary l with $\Im l \neq 0$, *denote by $\mathfrak{m}_a(l)$ or $\mathfrak{m}_b(l)$ the linear subspace of $\mathfrak{P}$ consisting of all points (f) such that $w(x,l,f)$ is square summable in $(a,c]$ or $[c,b)$, respectively.* Since it does not depend on the choice of c, $a < c < b$, whether $w(x,l,f)$ is square summable in $(a,c]$ (or $[c,b)$) or not, $\mathfrak{m}_a(l)$ *and $\mathfrak{m}_b(l)$ are independent of the choice of c. The linear spaces $\mathfrak{m}_a(l)$, $\mathfrak{m}_b(l)$ will be called main spaces with respect to the operator L.* From (2.3) follows

$$\text{(2.6)} \qquad w(x,\bar{l},\bar{f}) = \bar{w}(x,l,f),$$

whence we get

$$\text{(2.7)} \qquad \mathfrak{m}_a(\bar{l}) = \bar{\mathfrak{m}}_a(l), \qquad \mathfrak{m}_b(\bar{l}) = \bar{\mathfrak{m}}_b(l).$$

For arbitrary l with $\Im l \neq 0$, we put

$$h(f;x,l) = [w\bar{w}](x)/2i\Im l, \qquad (w = w(x,l,f)).$$

Obviously $h(f;x,l)$ is a hermitian form in f and, because of the Green's formula, we have

$$(2.8) \qquad h(f;x,l) - h(f;y,l) = \int_y^x |\, w(x,l,f)\,|^2\, dx, \qquad (\Im l \neq 0).$$

Hence, for fixed f and l, *$h(f;x,l)$ is a monotone increasing function of x.* Now we divide the space $\mathfrak{P}$ into three parts:

$$\mathfrak{P}^+(x,l) = \{(f)\,;\, h(f;x,l) > 0\},$$
$$\mathfrak{P}^0(x,l) = \{(f)\,;\, h(f;x,l) = 0\},$$
$$\mathfrak{P}^-(x,l) = \{(f)\,;\, h(f;x,l) < 0\}.$$

Evidently $\mathfrak{P}^+(x,l)$ and $\mathfrak{P}^-(x,l)$ are open subsets of $\mathfrak{P}$; $\mathfrak{P}^0(x,l)$ is a quadratic hypersurface, constituting the common boundary of $\mathfrak{P}^+(x,l)$ and $\mathfrak{P}^-(x,l)$. Denote in general the closure of a subset $\mathfrak{S}$ in $\mathfrak{P}$ by $[\mathfrak{S}]$. Since $h(f;x,l)$ is a monotone increasing function of x, $\mathfrak{P}^-(x,l)$ is monotone decreasing in the sense that $[\mathfrak{P}^-(x,l)] \subset \mathfrak{P}^-(y,l)$ for $x > y$; similarly we have $[\mathfrak{P}^+(y,l)] \subset \mathfrak{P}^+(x,l)$ for $x > y$. Putting $\mathfrak{k}_b(l) = \bigcap_x \mathfrak{P}^-(x,l)$, $\mathfrak{k}_a(l) = \bigcap_x \mathfrak{P}^+(x,l)$, we infer therefore that *$\mathfrak{k}_a(l)$ and $\mathfrak{k}_b(l)$ are compact, not empty, and have no common points.*[9] *Furthermore we have*

$$(2.9) \qquad \mathfrak{k}_a(l) \subset \mathfrak{m}_a(l), \qquad \mathfrak{k}_b(l) \subset \mathfrak{m}_b(l).$$

Indeed, if $f \in \mathfrak{k}_b(l)$, then f lies in all $\mathfrak{P}^-(x,l)$, $a < x < b$, and therefore, by (2.8),

$$\int_c^x |\, w(x,l,f)\,|^2\, dx + h(f;c,l) = h(f;x,l) < 0.$$

This shows clearly that $w(x,l,f)$ is square summable in $[c,b)$, proving that f is contained in $\mathfrak{m}_b(l)$; thus we get the second formula in (2.9). The first formula can be proved similarly. Again, from (2.6) follows

$$(2.10) \qquad h(f;x,\bar{l}) = h(f;x,l).$$

This yields immediately $\mathfrak{P}^{\pm}(x,\bar{l}) = \bar{\mathfrak{P}}^{\pm}(x,l)$ and consequently

$$(2.11) \qquad \mathfrak{k}_a(\bar{l}) = \bar{\mathfrak{k}}_a(l), \qquad \mathfrak{k}_b(\bar{l}) = \bar{\mathfrak{k}}_b(l).$$

Under the unimodular transformation (U) of the fundamental solutions mentioned above, the function $h(f;x,l)$ is invariant, whereas the sets $\mathfrak{P}^{\pm}(x,l)$ and their common parts $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$ are transformed according to the transformation (V) which is contragredient to (U).

For our purpose it is convenient to introduce the "*skew-product*"

$$[fg]_l = [w(l,f)w(l,g)] = \Sigma[s_j(l)s_k(l)] \cdot f^j g^k.$$

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[9] The sets $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$ correspond to the "limit circle" in the theory of H. Weyl. See Weyl, [11], pp. 221-231; [13], pp. 235-238.

DEFINITION 2.2. *By a null plane with respect to l will be meant a linear subspace $\mathfrak{p}$ of $\mathfrak{P}$ such that $[fg]_l = 0$ for all $f, g \varepsilon \mathfrak{p}$.*

In case the fundamental solutions $s_1, s_2, \cdots, s_n$ constitute a canonical system, the skew product becomes

$$[fg]_l = \sum_{j=1}^{\nu} (f^j g^{\nu+j} - f^{\nu+j} g^j),$$

so that, in this case, *the notion of "null plane" does not depend on l.*

THEOREM 2.1. *The sets $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$ contain respectively at least one $(\nu-1)$-dimensional null plane with respect to l.*

To show this we shall first prove

LEMMA 2.1. *Every $[\mathfrak{P}^+(y,l)]$, $a < y < b$, contains at least one $(\nu-1)$-dimensional null plane $\mathfrak{p}$.*

Proof. As the system of fundamental solutions we may choose without loss of generality the canonical system constituted of the solutions $s_j^0(x,l)$ $(j = 1, 2, \cdots, n)$ satisfying the *real* boundary conditions $[d^m s_j^0(x,l)/dx^m]_{x=y} = e_j^{(m)}$, $(m = 0, 1, \cdots, n-1;\ j = 1, \cdots, n)$ at the point y. Then, introducing a new system of coordinates:

$$\tilde{f}^j = (f^j + i f^{\nu+j})/\sqrt{2}, \qquad \tilde{f}^{\nu+j} = (f^j - i f^{\nu+j})/\sqrt{2}, \qquad (j = 1, \cdots, \nu),$$

we have

$$h(f; y, l) = \sum_{j=1}^{\nu} (|\tilde{f}^j|^2 - |\tilde{f}^{\nu+j}|^2)/2\Im l, \qquad [fg]_l = i \sum_{j=1}^{\nu} (\tilde{f}^j \bar{\tilde{g}}^{\nu+j} - \tilde{f}^{\nu+j} \bar{\tilde{g}}^j).$$

Now, assuming that $\Im l > 0$ (the case: $\Im l < 0$ can be treated similarly), it is obvious by the above formulae that the linear subspace $\mathfrak{p}$ consisting of all points f of the form $(\tilde{f}^1, \cdots, \tilde{f}^\nu, 0, \cdots, 0)$, $\tilde{f}^j$ being arbitrary complex numbers, is a $(\nu-1)$-dimensional null plane contained in $[\mathfrak{P}^+(y,l)]$.

Proof of Theorem 2.1. In order to prove the existence of a $(\nu-1)$-dimensional null plane $\mathfrak{p}$ contained in $\mathfrak{k}_a(l)$, we denote $(\nu-1)$-dimensional null planes generally by $\mathfrak{p}$ and the set consisting of all $(\nu-1)$-dimensional null planes by $\mathfrak{N}$. Evidently $\mathfrak{N}$ constitutes an algebraic variety without singularity in the $[C_\nu^n - 1]$-dimensional projective space. $\mathfrak{N}$ is therefore compact in the natural topology. Putting $\mathfrak{N}^+(x) = \{\mathfrak{p}; \mathfrak{p} \subset [\mathfrak{P}^+(x,l)]\}$, we have

$$\{\mathfrak{p}; \mathfrak{p} \subset \mathfrak{k}_a(l)\} = \bigcap_x \mathfrak{N}^+(x), \tag{2.12}$$

since $\mathfrak{k}_a(l)$ is the common part of all $[\mathfrak{P}^+(x,l)]$, $a < x < b$. On the other

hand, $\mathfrak{N}^+(x)$ is obviously a *compact* subset of $\mathfrak{N}$ and, as Lemma 2.1 shows, *not empty*; furthermore $\mathfrak{N}^+(x)$ *decreases monotonously when x decreases,* since $\mathfrak{P}^+(x,l)$ decreases monotonously. Hence *the intersection* $\bigcap_x \mathfrak{N}^+(x)$ *is not empty.* This shows, combined with (2.12), that there exists at least one null plane $\mathfrak{p}$ contained in $\mathfrak{k}_a(l)$. As to $\mathfrak{k}_b(l)$ the proof will be accomplished in the same way.

From Theorem 2.1 thus proved we can deduce some important conclusions concerning the main spaces $\mathfrak{m}_a(l)$, $\mathfrak{m}_b(l)$. Let $\mathfrak{p}_a$, $\mathfrak{p}_b$ be $(\nu-1)$-dimensional null planes with respect to l contained respectively in $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$. Then, since $\mathfrak{k}_a(l) \subset \mathfrak{m}_a(l)$, $\mathfrak{k}_b(l) \subset \mathfrak{m}_b(l)$, we get first $\dim \mathfrak{m}_a(l) \geqq \nu - 1$, $\dim \mathfrak{m}_b(l) \geqq \nu - 1$. Furthermore $\dim \mathfrak{m}_a(l)$ *and* $\dim \mathfrak{m}_b(l)$ *do not depend on the parameter l, provided $\Im l \neq 0$,* as will be proved in **3** below (see Theorem 3.2). Putting

$$(2.13) \qquad \tau_a = \dim \mathfrak{m}_a(l) - \nu + 1, \qquad \tau_b = \dim \mathfrak{m}_b(l) - \nu + 1 \quad (\Im l \neq 0),$$

we obtain therefore two non-negative integers τ_a, τ_b which are characteristic for L.

DEFINITION 2.3. *L will be called a differential operator of the type (τ_a, τ_b), τ_a, τ_b being the non-negative integers defined by* (2.13). *The sum $\tau = \tau_a + \tau_b$ will be called the excess index of L.*

Possible values of τ_a, τ_b are obviously $0, 1, 2, \cdots, \nu$; thus *formally self-adjoint differential operators L of the order $n = 2\nu$ can be classified into $(\nu+1)^2$ different types.* This classification is a generalization of H. Weyl's one[10] concerning the differential operators of the second order.

For a later purpose we introduce the space $\mathfrak{m}(l) = \mathfrak{m}_a(l) \cap \mathfrak{m}_b(l)$. Since $\mathfrak{k}_a(l)$ and $\mathfrak{k}_b(l)$ have no common point, $\mathfrak{p}_a$ and $\mathfrak{p}_b$ have also no common point, and therefore $\mathfrak{m}_a(l) + \mathfrak{m}_b(l) \supseteq \mathfrak{p}_a + \mathfrak{p}_b = \mathfrak{P}$.[11] Hence we get

$$(2.14) \qquad \dim \mathfrak{m}(l) = \tau - 1.$$

The space $\mathfrak{m}(l)$ is clearly the set of all points (f) such that $w(x,l,f)$ is square summable in (a,b). *The excess index τ is therefore equal to the number of linearly independent solutions of the differential equation $L[u] = l \cdot u$ which are square summable in (a,b), provided $\Im l \neq 0$.*

3. Green's function. Let $\mathfrak{H}$ be the Hilbert space consisting of all

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[10] Weyl, [11], pp. 221-231; [12], p. 443; [13].

[11] "$+$" means the "join" in the sense of the projective geometry.

square summable functions in the interval (a, b); the inner product of the functions u, v in $\mathfrak{H}$ will be denoted by (u, v), the norm of u by $\| u \|$.

DEFINITION 3.1. *We denote by $\mathfrak{D}$ the linear subspace of $\mathfrak{H}$ constituted of all functions $u(x)$ satisfying the following four conditions:*

i) *$u(x)$ is square summable in (a, b),*

ii) *in the open interval (a, b), $u(x)$ admits continuous derivatives up to the order $n-1$,*

iii) *$(n-1)$-th derivative $u^{(n-1)}(x)$ is absolutely continuous in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$ (so that $L[u]$ can be defined for almost all x, $a < x < b$),*

iv) *$L[u]$ is square summable in (a, b).*

Considered as a linear operator having $\mathfrak{D}$ as its domain, the differential operator L will be denoted by T, i. e. *we put*

$$Tu = L[u], \qquad \text{for } u \in \mathfrak{D}. \tag{3.1}$$

Obviously the bilinear form $[uv](x)$ can be defined for two arbitrary functions u, v in $\mathfrak{D}$ so far as $a < x < b$. Furthermore we infer from Green's formula that, *for arbitrary u, $v \in \mathfrak{D}$, the limits $[uv](a) = \lim_{x \to a} [uv](x)$, $[uv](b) = \lim_{x \to b} [uv](x)$ exist.*

Now we shall introduce *Green's function*. Let l be a fixed complex number with $\Im l \neq 0$ and $\mathfrak{p}_a$, $\mathfrak{p}_b$ be two $(\nu - 1)$-dimensional null planes with respect to l contained respectively in $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$. Then, choosing n linearly independent vectors $f_1, f_2, \cdots, f_\nu, f_{\nu+1}, \cdots, f_n$ such that $f_\alpha \in \mathfrak{p}_a$ (for $\alpha = 1, 2, \cdots, \nu$), $f_\beta \in \mathfrak{p}_b$ (for $\beta = \nu + 1, \cdots, n$) arbitrarily, we put

$$\begin{aligned} G(x, y; l, \mathfrak{p}_a, \mathfrak{p}_b) &= G(y, x; l, \mathfrak{p}_a, \mathfrak{p}_b) \\ &= \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l) w_\beta(x) w_\alpha(y), \qquad (x \geqq y), \end{aligned} \tag{3.2}$$

where $w_\gamma(x) = w(x, l, f_\gamma)$ $(\gamma = 1, \cdots, \nu, \nu + 1, \cdots, n)$ and $(F_{\alpha\beta}(l))$ means the inverse matrix of $([f_\alpha f_\beta]_l)$. The function $G(x, y; l, \mathfrak{p}_a, \mathfrak{p}_b)$ thus defined will be called *Green's function*. The matrix $([f_\alpha f_\beta]_l)$ is skew-symmetric and, since $f_1, \cdots, f_\nu$ or $f_{\nu+1}, \cdots, f_n$ lie on the null plane $\mathfrak{p}_a$ or $\mathfrak{p}_b$, respectively, we have

$$[f_\alpha f_\beta]_l = 0, \qquad \text{for } \alpha \leqq \nu, \beta \leqq \nu, \text{ and for } \alpha > \nu, \beta > \nu. \tag{3.3}$$

Consequently we see

$$F_{\alpha\beta}(l) = -F_{\beta\alpha}(l), \tag{3.4}$$

$$(3.5)\qquad F_{\alpha\beta}(l)=0, \qquad \text{for } \alpha\leqq\nu, \beta\leqq\nu, \text{ and for } \alpha>\nu, \beta>\nu.$$

Introduce now the important matrix

$$(3.6)\qquad M^{jk}(l,\mathfrak{p}_a,\mathfrak{p}_b)=\sum_{\alpha=1}^{\nu}\sum_{\beta=\nu+1}^{n}F_{\alpha\beta}(l)f_\beta{}^j f_\alpha{}^k.$$

Then we obtain, using (3.3) and (3.5),

$$\sum_k\sum_m M^{mj}(l,\mathfrak{p}_a,\mathfrak{p}_b)[s_m(l)s_k(l)]\cdot f_\gamma{}^k=\begin{cases} f_\gamma{}^j, & (1\leqq\gamma\leqq\nu),\\ 0, & (\nu<\gamma\leqq n),\end{cases}$$

proving the relation

$$(3.7)\qquad \sum_k\sum_m M^{mj}(l,\mathfrak{p}_a,\mathfrak{p}_b)[s_m(l)s_k(l)]\cdot f^k=\begin{cases} f^j, & (f\,\varepsilon\,\mathfrak{p}_a),\\ 0, & (f\,\varepsilon\,\mathfrak{p}_b).\end{cases}$$

Similarly we get

$$(3.8)\qquad \sum_k\sum_m M^{jm}(l,\mathfrak{p}_a,\mathfrak{p}_b)[s_m(l)s_k(l)]\cdot f^k=\begin{cases} 0, & (f\,\varepsilon\,\mathfrak{p}_a),\\ -f^j, & (f\,\varepsilon\,\mathfrak{p}_b).\end{cases}$$

These relations show that $M^{jk}(l,\mathfrak{p}_a,\mathfrak{p}_b)$ *is determined uniquely by* l, $\mathfrak{p}_a$, $\mathfrak{p}_b$ *and does not depend on the choice of the points* $f_1,\cdots,f_\nu$ *on* $\mathfrak{p}_a$ *and* $f_{\nu+1},\cdots,f_n$ *on* $\mathfrak{p}_b$. By using the matrix $M^{jk}(l,\mathfrak{p}_a,\mathfrak{p}_b)$, Green's function can be rewritten as

$$(3.9)\qquad G(x,y;l,\mathfrak{p}_a,\mathfrak{p}_b)=\sum M^{jk}(l,\mathfrak{p}_a,\mathfrak{p}_b)s_j(x,l)s_k(y,l), \qquad (x\geqq y).$$

Hence Green's function is also determined uniquely by l, $\mathfrak{p}_a$, $\mathfrak{p}_b$ *and does not depend on the choice of* $f_1,\cdots,f_\nu,\cdots,f_n$ *appearing in its definition.* Incidentally, we infer from (3.7) and (3.8) the relation

$$(3.10)\qquad \sum_m\{M^{mj}(l,\mathfrak{p}_a,\mathfrak{p}_b)-M^{jm}(l,\mathfrak{p}_a,\mathfrak{p}_b)\}[s_m(l)s_k(l)]=\delta_k{}^j.$$

DEFINITION 3.2. *$G(l,\mathfrak{p}_a,\mathfrak{p}_b)$ will denote the integral operator with the kernel $G(x,y;l,\mathfrak{p}_a,\mathfrak{p}_b)$, that is the operator defined by*

$$(3.11)\qquad G(l,\mathfrak{p}_a,\mathfrak{p}_b)v(x)=\int_a^b G(x,y;l,\mathfrak{p}_a,\mathfrak{p}_b)v(y)\,dy$$
$$=\sum_{\alpha=1}^{\nu}\sum_{\beta=\nu+1}^{n}F_{\alpha\beta}(l)\{w_\beta(x)\int_a^x w_\alpha(y)v(y)dy+w_\alpha(x)\int_x^b w_\beta(y)v(y)dy\},$$

$v(x)$ being an arbitrary function in $\mathfrak{H}$.

Since $f_\alpha\,\varepsilon\,\mathfrak{p}_a\subset\mathfrak{k}_a(l)\subset\mathfrak{m}_a(l)$ $(\alpha=1,\cdots,\nu)$, the functions $w_\alpha(x)=w(x,l,f_\alpha)$ $(\alpha=1,2,\cdots,\nu)$ are square summable in $(a,x]$, and similarly the functions $w_\beta(x)$ $(\beta=\nu+1,\cdots,n)$ are square summable in $[x,b)$. Hence *the integrals in* (3.11) *converge absolutely*; thus the function $u(x)=G(l,\mathfrak{p}_a,\mathfrak{p}_b)v(x)$ is defined for all $v(x)$ in $\mathfrak{H}$. Now we *shall prove*

that the function $u(x)$ thus defined admits continuous derivatives up to the order $n-1$, that the $(n-1)$-th derivative $u^{(n-1)}(x)$ is absolutely continuous in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$, and that $u(x)$ satisfies the differential equation:

$$L[u] - l \cdot u = v. \tag{3.12}$$

From the relation

$$[f_\alpha f_\beta]_l = [w_\alpha w_\beta] = \sum_{j+k \leqq n-1} B_{jk}(x) w_\alpha{}^{(j)}(x) w_\beta{}^{(k)}(x)$$

we get readily

$$\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} F_{\alpha\beta}(l) w_\alpha{}^{(j)}(x) w_\beta{}^{(k)}(x) = B^{-1}{}_{jk}(x), \tag{3.13}$$

where $(B^{-1}{}_{jk})$ means the inverse matrix of (B_{jk}). Since

$$B_{jk}(x) = (-1)^{j+1} p_0(x) \ (\text{for } j+k = n-1), \ = 0 \ (\text{for } j+k > n-1), \tag{3.14}$$

we have

$$B^{-1}{}_{jk}(x) = (-1)^j / p_0(x) \ (\text{for } j+k = n-1), \ = 0 \ (\text{for } j+k < n-1).$$

Combined with (3.4) and (3.5), (3.13) yields therefore

$$\sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l) \{ w_\beta{}^{(m)}(x) w_\alpha(x) - w_\alpha{}^{(m)}(x) w_\beta(x) \} = 0, \quad (m = 0, 1, \cdots, n-2), \tag{3.15}$$

$$\sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l) \{ w_\beta{}^{(n-1)}(x) w_\alpha(x) - w_\alpha{}^{(n-1)}(x) w_\beta(x) \} = 1/p_0(x). \tag{3.16}$$

Using (3.15), we infer readily that $u(x)$ is differentiable in (a, b) up to the order $n-1$ and that the m-th derivatives $u^{(m)}(x)$ are given by

$$u^{(m)}(x) = \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l) \{ w_\beta{}^{(m)}(x) \int_a^x w_\alpha v \, dy + w_\alpha{}^{(m)}(x) \int_x^b w_\beta v \, dy \}, \quad (m = 1, 2, \cdots, n-1). \tag{3.17}$$

Evidently $u^{(m)}(x)$ $(m = 0, \cdots, n-1)$ are absolutely continuous in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$. Especially $u^{(n-1)}(x)$ is differentiable almost everywhere in (a, b) and, as one readily infers by (3.16), the derivative $u^{(n)}(x)$ is given by

$$u^{(n)}(x) = v(x)/p_0(x) + \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l) \{ w_\beta{}^{(n)}(x) \int_a^x w_\alpha v \, dy + w_\alpha{}^{(n)}(x) \int_x^b w_\beta v \, dy \}.$$

Combined with (3.17), this shows clearly that $u(x)$ satisfies the differential equation (3.12).

Incidentally, from (3.14), (3.15) and (3.16) follows the identity

$$(3.18)\qquad \sum_{\alpha=1}^{\nu}\sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l)\{[w_\beta v](x)w_\alpha(x)-[w_\alpha v](x)w_\beta(x)\}=v(x),$$

which will be used in **5** below.

THEOREM 3.1. *$G(l,\mathfrak{p}_a,\mathfrak{p}_b)$ is a bounded linear operator whose norm does not exceed $|\Im l|^{-1}$, provided $\Im l\neq 0$. Furthermore, for an arbitrary function $v(x)$ in $\mathfrak{H}$, the function $u(x)=G(l,\mathfrak{p}_a,\mathfrak{p}_b)v(x)$ belongs to $\mathfrak{D}$ and satisfies the differential equation $L[u]-l\cdot u=v$.*

Proof.[12] To show that $G(l,\mathfrak{p}_a,\mathfrak{p}_b)$ is a bounded operator, we take two points t, z such that $a<t<z<b$ arbitrarily and put

$$u_1(x)=\sum_{\alpha=1}^{\nu}\sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l)\{w_\beta(x)\int_t^x w_\alpha v\,dy+w_\alpha(x)\int_x^z w_\beta v\,dy\}.$$

Then $u_1(x)$ satisfies also the differential equation $L[u_1]-l\cdot u_1=v$. Hence we get, using Green's formula,

$$(3.19)\qquad \int_t^z|u_1|^2dx+(1/2i\Im l)\int_t^z(v\bar{u}_1-u_1\bar{v})\,dx$$
$$=\{[u_1\bar{u}_1](z)-[u_1\bar{u}_1](t)\}/2i\Im l.$$

Now, as one readily infers by (3.15), $u_1^{(m)}(z)$ $(m=0,1,\cdots,n-1)$ are given by $u_1^{(m)}(z)=w^{(m)}(z,l,g)$, where

$$g=\sum_{\beta=\nu+1}^{n}\{\sum_{\alpha=1}^{\nu}F_{\alpha\beta}(l)\cdot\int_t^z w_\alpha v\,dy\}\cdot f_\beta.$$

We have therefore

$$[u_1\bar{u}_1](z)/2i\Im l=h(g;z,l).$$

On the other hand, since f_β $(\beta=\nu+1,\cdots,n)$ lie on $\mathfrak{p}_b$, g lies also in $\mathfrak{p}_b$, while we have $\mathfrak{p}_b\subset\mathfrak{k}_b(l)\subset\mathfrak{P}^-(z,l)$. Hence $h(g;z,l)$ is negative and therefore $[u_1\bar{u}_1](z)/2i\Im l<0$; similarly we get $[u_1\bar{u}_1](t)/2i\Im l>0$. Using these inequalities, we conclude from (3.19) the inequality

$$\int_t^z|u_1|^2dx\leqq|\Im l|^{-2}\int_t^z|v|^2dx.$$

Now, making here $t\to a$, $z\to b$, we infer from this the inequality

$$(3.20)\qquad \|u\|\leqq|\Im l|^{-1}\|v\|,\qquad (u=G(l,\mathfrak{p}_a,\mathfrak{p}_b)v),$$

[12] We follow the method of H. Weyl. See Weyl, [11], pp. 228-231.

proving that $G(l, \mathfrak{p}_a, \mathfrak{p}_b)$ is a bounded operator and that its norm satisfies the inequality

$$|||\, G(l, \mathfrak{p}_a, \mathfrak{p}_b)\,||| \leqq |\, \Im l\,|^{-1}. \tag{3.21}$$

As was already proved, the function $u(x)$ satisfies the differential equation (3.12) and the conditions ii) and iii) in Definition 3.1, while (3.20) shows that $u(x)$ satisfies the condition i) in Definition 3.1. Now, using these results, we get

$$\| L[u] \| = \| l\cdot u + v \| \leqq (1 + |\, l\,| / |\, \Im l\,|) \cdot \| v \|,$$

proving that u satisfies also the condition iv) in Definition 3.1. Thus we see that the function $u(x)$ belongs to $\mathfrak{D}$.

COROLLARY. *We have*

$$(T - l)\, G(l, \mathfrak{p}_a, \mathfrak{p}_b) = 1, \qquad (\Im l \neq 0). \tag{3.22}$$

By virtue of Theorem 3.1, we can now prove the following theorem cited already in **2** above:

THEOREM 3.2. $\operatorname{Dim} \mathfrak{m}_a(l)$ *and* $\dim \mathfrak{m}_b(l)$ *do not depend on* l, *provided* $\Im l \neq 0$.

To prove this, we first introduce

DEFINITION 3.3. *For arbitrary* l *with* $\Im l \neq 0$, *we denote by* $\mathfrak{E}(l)$ *the eigenspace of the operator* T *corresponding to the eigenvalue* l.

Evidently $\mathfrak{E}(l)$ can be written as

$$\mathfrak{E}(l) = \{w(x, l, f)\,;\ (f) \ \varepsilon\ \mathfrak{m}(l) \text{ or } f = 0\}. \tag{3.23}$$

Combined with (2.14), this yields

$$\dim \mathfrak{E}(l) = \tau, \tag{3.24}$$

τ being the excess index of L. We have now to prove that τ is independent on l.[13]

Choose an arbitrary complex number l^0 with $\Im l^0 \neq 0$, put, for simplicity's sake, $G^0 = G(l^0, \mathfrak{p}_a{}^0, \mathfrak{p}_b{}^0)$, $\mathfrak{p}_a{}^0$, $\mathfrak{p}_b{}^0$ being $(\nu - 1)$-dimensional null planes with respect to l^0 contained respectively in $\mathfrak{k}_a(l^0)$, $\mathfrak{k}_b(l^0)$, and introduce the operator $K(l) = 1 - (l - l^0)\, G^0$. Again we denote by $C(l^0)$ the circular region in the l-plane consisting of all l satisfying $|\, l - l^0\,| < |\, \Im l^0\,|$. Then

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[13] We follow the method of H. Weyl. See Weyl, [13], pp. 238-247. Cf. also Stone, [7], Theorem 9.8.

we infer by (3.21) that, for arbitrary l in $C(l^0)$, the operator $K(l)$ has the inverse $K(l)^{-1}$. Indeed the inverse is given by the power series

$$K(l)^{-1} = 1 + \sum_{m=1}^{\infty} (l - l^0)^m (G^0)^m, \tag{3.25}$$

converging for $l \,\varepsilon\, C(l^0)$ in the sense of the norm: $|||\quad|||$. Thus in case $l \,\varepsilon\, C(l^0)$, $K(l)$ is a one-to-one transformation mapping $\mathfrak{H}$ on itself. Now, using (3.22), we get readily $(T - l^0) \cdot K(l)u = (T - l) \cdot u$, (for all $u \,\varepsilon\, \mathfrak{H}$). This shows that $K(l)$ *maps* $\mathfrak{E}(l)$ *on* $\mathfrak{E}(l^0)$ *one-to-one.* Hence $\dim \mathfrak{E}(l)$ is constant in $C(l^0)$, while, by choosing l^0 suitably, an arbitrary compact domain D in the half plane $\Im l > 0$ (or $\Im l < 0$) can be covered by $C(l^0)$. Consequently $\dim \mathfrak{E}(l)$ is constant in each half plane $\Im l > 0$, $\Im l < 0$. On the other hand, from (2.6) and (3.23) follows $\mathfrak{E}(\bar{l}) = \bar{\mathfrak{E}}(l)$. Hence $\dim \mathfrak{E}(l)$ *does not depend on* l, provided $\Im l \neq 0$.

To deduce Theorem 3.2 from this result, we choose a fixed point c, $a < c < b$, arbitrarily, and *observe the operator* L *in the interval* $(a, c]$ *instead of* (a, b). *The meanings of the notations* $\mathfrak{H}$, $\mathfrak{m}(l)$, $\mathfrak{E}(l)$, *etc. are to be changed correspondingly, e. g.* $\mathfrak{H}$ *denotes the Hilbert space consisting of all square summable functions defined in* $(a, c]$, $\mathfrak{m}(l)$ *the subspace of* $\mathfrak{P}$ *consisting of all* f *such that* $w(x, l, f)$ *is square summable in* $(a, c]$, etc. Then, since every solution $w(x, l, f)$ is continuous in c, we have $\mathfrak{m}_c(l) = \mathfrak{P}$ and therefore $\mathfrak{m}(l) = \mathfrak{m}_a(l)$, while, as was proved above, $\dim \mathfrak{m}(l) = \dim \mathfrak{E}(l) - 1$ does not depend on l. Hence $\dim \mathfrak{m}_a(l)$ is independent on l. It can be proved similarly that $\dim \mathfrak{m}_b(l)$ does not depend on l.

In this connection, we shall prove furthermore

THEOREM 3.3. *As functions of* l, *the main spaces* $\mathfrak{m}_a(l)$, $\mathfrak{m}_b(l)$, *and their intersection* $\mathfrak{m}(l)$ *are holomorphic, provided* $\Im l \neq 0$, *in the sense that, for every* l_0 *with* $\Im l_0 \neq 0$, *the suitably normalized*[14] *Plücker coordinates* $\mathfrak{m}_a^{ij\ldots k}(l)$, $\mathfrak{m}_b^{ij\ldots k}(l)$, $\mathfrak{m}^{ij\ldots k}(l)$ *of* $\mathfrak{m}_a(l)$, $\mathfrak{m}_b(l)$, $\mathfrak{m}(l)$ *are holomorphic with respect to* l *in some neighborhood of* l_0.

To prove this,[15] we introduce

DEFINITION 3.4. *Let* $r(x, l)$ *be a square summable function of* x *defined in* (a, b) *depending on the parameter* l *and* D *be a domain in the* l*-plane.*

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[14] Generally, the homogeneous coordinates of a geometric object are said to be *normalized,* if one of the coordinates is equal to 1.

[15] We follow the method of H. Weyl. See Weyl, [13], pp. 238-247. I am indebted to Prof. Weyl for suggesting to me to apply here his direct method.

Then $r(x,l)$ is said to be holomorphic with respect to l in D in the sense of the norm: $\| \quad \|$, if, for every point l_0 in D, $r(x,l)$ can be expanded into the power series

$$r(x,l) = \sum_{m=0}^{\infty} r_m(x)(l-l_0)^m, \qquad (r_m(x) \,\varepsilon\, \mathfrak{H})$$

converging in the sense of the norm: $\| \quad \|$ in some neighborhood of l_0.

LEMMA 3.1. *Let a compact domain D in the half-plane $\Im l > 0$ (or $\Im l < 0$) be given. Then we can choose a base $\{r_\sigma(x,l); \sigma = 1, 2, \cdots, \tau\}$ of the functional space $\mathfrak{E}(l)$ so that $r_\sigma(x,l)$ are holomorphic in D with respect to l in the sense of the norm: $\| \quad \|$ and that, for every fixed x, $a < x < b$, the derivatives $d^m r_\sigma(x,l)/dx^m$ $(m = 0, 1, \cdots, n-1)$ are holomorphic in D with respect to l (in the usual sense).*

Proof. As was proved above, we have $\mathfrak{E}(l) = K(l)^{-1}\mathfrak{E}(l^0)$ for $l \,\varepsilon\, C(l^0)$. Hence, choosing a base $\{r^0{}_\sigma(x); \sigma = 1, \cdots, \tau\}$ of the space $\mathfrak{E}(l^0)$ arbitrarily and putting

$$r_\sigma(x,l) = K(l)^{-1} r^0{}_\sigma(x), \tag{3.26}$$

we obtain, for arbitrary $l \,\varepsilon\, C(l^0)$, the base $\{r_\sigma(x,l); \sigma = 1, \cdots, \tau\}$ of the space $\mathfrak{E}(l)$. Now it follows from (3.25) that $r_\sigma(x,l)$ are holomorphic in $C(l^0)$ with respect to l in the sense of the norm: $\| \quad \|$. Again we get from (3.26) $r_\sigma(x,l) = r^0{}_\sigma(x) + (l - l^0) G^0 \cdot r_\sigma(x,l)$, which yields, combined with (3.17),

$$d^m r_\sigma(x,l)/dx^m = d^m r^0{}_\sigma(x)/dx^m + (l - l^0) \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l^0)$$

$$\cdot \{(d^m w^0{}_\beta/dx^m) \cdot \int_a^x w^0{}_\alpha(y) r_\sigma(y,l)\,dy + (d^m w^0{}_\alpha/dx^m)$$

$$\cdot \int_x^b w^0{}_\beta(y) r_\sigma(y,l)\,dy\}, \qquad (m = 0, 1, \cdots, n-1).$$

Whence we infer that $d^m r_\sigma(x,l)/dx^m$ $(m = 0, 1, \cdots, n-1)$ are also holomorphic with respect to l in $C(l^0)$. This proves Lemma 3.1, since, by a suitable choice of l^0, D can be covered by $C(l^0)$.

Proof of Theorem 3.3. Let D be an arbitrary compact domain in the half plane $\Im l > 0$ (or $\Im l < 0$) and $\{r_\sigma(x,l); \sigma = 1, 2, \cdots, \tau\}$ be the base of $\mathfrak{E}(l)$ mentioned in Lemma 3.1 above. Then, putting

$$r_\sigma(x,l) = \sum_{j=1}^{n} s_j(x,l) \cdot f_\sigma{}^j(l), \qquad (\sigma = 1, 2, \cdots, \tau),$$

we obtain τ linearly independent points $f_\sigma(l)$ $(\sigma = 1, 2, \cdots, \tau)$ generating

the linear space $\mathfrak{m}(l)$. The coordinates $f_\sigma^j(l)$ are determined by the system of linear equations

$$\sum_{j=1}^{n} s_j^{(m)}(x,l)\cdot f_\sigma^j(l)=r_\sigma^{(m)}(x,l), \qquad (m=0,1,\cdots,n-1)$$

for a fixed point x, while $s_j^{(m)}(x,l)$, $r_\sigma^{(m)}(x,l)$ are holomorphic with respect to l in D and $[s_1s_2\cdots s_n]=1$. Hence $f_\sigma^j(l)$ are also holomorphic in D, proving that the linear space $\mathfrak{m}(l)$ depends on l holomorphically, provided $\Im l\neq 0$.

Consider now the operator L in the interval $(a,c]$, $a<c<b$, instead of (a,b), as in the proof of Theorem 3.2. Then $\mathfrak{m}_a(l)$ coincides with $\mathfrak{m}(l)$. Hence $\mathfrak{m}_a(l)$ is holomorphic with respect to l, and similarly $\mathfrak{m}_b(l)$ depends also holomorphically on l, provided $\Im l\neq 0$.

Now we turn to investigate the operator T. First we prove

THEOREM 3.4. *T is a closed operator. The domain $\mathfrak{D}$ of T consists of all elements $u\,\varepsilon\,\mathfrak{H}$ having the following form:*

$$u=G(l,\mathfrak{p}_a,\mathfrak{p}_b)v+w, \qquad (v\,\varepsilon\,\mathfrak{H},\ w\,\varepsilon\,\mathfrak{E}(l)), \tag{3.27}$$

l being a fixed complex number with $\Im l\neq 0$.[16]

Proof. By virtue of Theorem 3.1, it is obvious that every u of the form (3.27) belongs to $\mathfrak{D}$. Conversely, let an arbitrary element $u\,\varepsilon\,\mathfrak{D}$ be given. Then, putting

$$v=(T-l)u, \qquad w=u-G(l,\mathfrak{p}_a,\mathfrak{p}_b)v,$$

we infer, by (3.22), $(T-l)w=0$, proving that w belongs to $\mathfrak{E}(l)$. Thus we see that u has the form (3.27). To prove that T is a closed operator, consider a sequence $\{u_m\}$, $u_m\,\varepsilon\,\mathfrak{D}$, such that the limits $u=\lim u_m$, $u^*=\lim Tu_m$, exist. As was proved above, u_m can be written as

$$u_m=G(l,\mathfrak{p}_a,\mathfrak{p}_b)(Tu_m-lu_m)+w_m, \qquad (w_m\,\varepsilon\,\mathfrak{E}(l)).$$

Hence there exists the limit $w=\lim w_m$ and we have

$$u=G(l,\mathfrak{p}_a,\mathfrak{p}_b)(u^*-lu)+w, \qquad (w\,\varepsilon\,\mathfrak{E}(l)), \tag{3.28}$$

proving that u belongs to $\mathfrak{D}$. Again, using (3.22), we get, from (3.28), $(T-l)u=u^*-lu$, or $Tu=u^*$; thus T is a closed operator, q. e. d.

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[16] The closures and adjoints of ordinary differential operators were determined by I. Halperin under some restrictions imposed on the coefficients of the differential operators. See Halperin, [2]. The operator L under consideration is not necessarily subject to Halperin's restrictions.

It is to be noted here that, *for the element u having the form* (3.27) we have $(T-l)u=v$, as one readily infers by (3.22).

It is obvious that the *adjoint operator* $G^*(l,\mathfrak{p}_a,\mathfrak{p}_b)$ of $G(l,\mathfrak{p}_a,\mathfrak{p}_b)$ is given by

$$(3.29)\qquad G^*(l,\mathfrak{p}_a,\mathfrak{p}_b)v(x)=\int_a^b\bar{G}(x,y;l,\mathfrak{p}_a,\mathfrak{p}_b)v(y)dy.$$

Now, using this, we shall determine the *adjoint operator of T, which will be denoted by T_0.*

THEOREM 3.5. *The domain $\mathfrak{D}_0$ of the adjoint operator T_0 of T consists of all elements $u\,\varepsilon\,\mathfrak{D}$ having the following form*:

$$(3.30)\qquad u=G(l,\mathfrak{p}_a,\mathfrak{p}_b)v,\qquad v\,\varepsilon\,\mathfrak{H}-\bar{\mathfrak{E}}(l)^{17};$$

thus $\mathfrak{D}_0$ is a subspace of $\mathfrak{D}$. The adjoint operator T_0 is obtained from T by restricting the domain $\mathfrak{D}$ of T to $\mathfrak{D}_0$. T_0 is a symmetric operator.

Proof. The relation: $u^*=T_0u$ means, by definition, that

$$(3.31)\qquad (u,Tu_1)=(u^*,u_1)$$

holds for every element $u_1\,\varepsilon\,\mathfrak{D}$. Obviously the domain $\mathfrak{D}$ remains unchanged by the conjugation $u(x)\rightarrow\overline{u(x)}$. Hence, using (3.29), we infer by Theorem 3.4 that $\mathfrak{D}$ consists of all elements of the form $u_1=G^*(l,\mathfrak{p}_a,\mathfrak{p}_b)v_1+\bar{w}$, $(v_1\,\varepsilon\,\mathfrak{H},\ w\,\varepsilon\,\mathfrak{E}(l))$. By inserting this expression in (3.31), the condition (3.31) becomes $(u+lGu-Gu^*,v_1)+(lu-u^*,\bar{w})=0$, where $G=G(l,\mathfrak{p}_a,\mathfrak{p}_b)$. (3.31) is therefore equivalent to

$$(3.32)\qquad \begin{cases} u=G(l,\mathfrak{p}_a,\mathfrak{p}_b)(u^*-lu), \\ (u^*-lu,\bar{w})=0, \qquad \text{for all } w\,\varepsilon\,\mathfrak{E}(l). \end{cases}$$

Now it is obvious that every u satisfying (3.32) has the form (3.30). Assume conversely u to have the form (3.30). Then putting $u^*=v+lu$, we get immediately (3.32). Thus $\mathfrak{D}_0$ consists of all elements u of the form (3.30). Furthermore, for such u, we get, using (3.22) and (3.32), $Tu=u^*=T_0u$. Thus T is an extension of T_0, q. e. d.

It is to be noted here that *the deficiency index*[18] *of the symmetric operator T_0* is (τ,τ).

Finally we shall prove

THEOREM 3.6.[19] *For arbitrary fixed l with $\Im l\neq0$, the domain $\mathfrak{D}$ is decomposed as a linear space into the direct sum*:

[17] $\mathfrak{H}-\mathfrak{E}$ means the orthogonal complement of the subspace $\mathfrak{E}$ in $\mathfrak{H}$.

[18] See Stone, [7], Chap. IX.

[19] This theorem is essentially reduced to Theorem 9.4 in Stone, [7]. Here we shall give another proof based on Green's function.

$$\mathfrak{D} = \mathfrak{D}_0 + \mathfrak{E}(l) + \bar{\mathfrak{E}}(l), \qquad (\Im l \neq 0). \tag{3.33}$$

Proof. It is sufficient to show that every element $u \varepsilon \mathfrak{D}$ is decomposed uniquely in the form

$$u = u_0 + w_1 + \bar{w}_2, \qquad (u_0 \varepsilon \mathfrak{D}_0, w_1, w_2 \varepsilon \mathfrak{E}(l)). \tag{3.34}$$

By Theorem 3.4, u is represented as $u = Gv + w_3$ $(w_3 \varepsilon \mathfrak{E}(l))$, where G means $G(l, \mathfrak{p}_a, \mathfrak{p}_b)$. Decompose now v into two parts: $v = v_0 + \bar{w}$, $(v_0 \varepsilon H - \bar{\mathfrak{E}}(l), w \varepsilon \mathfrak{E}(l))$ and put $u_0 = Gv_0$, $w_1 = G\bar{w} + \bar{w}/2i\Im l + w_3$, $w_2 = w/2i\Im l$. Then we have the desired decomposition $u = u_0 + w_1 + \bar{w}_2$, $(w_1, w_2 \varepsilon \mathfrak{E}(l))$.

To prove the uniqueness of the decomposition, it is sufficient to show that the relation

$$0 = u_0 + w_1 + \bar{w}_2, \qquad (u_0 \varepsilon \mathfrak{D}, w_1, w_2 \varepsilon \mathfrak{E}(l)) \tag{3.35}$$

implies $u_0 = w_1 = w_2 = 0$. Assuming (3.35), we have $2i\Im l \cdot (w_1, w_1) = (w_1, T_0 u_0) - (Tw_1, u_0) = 0$, proving that w_1 is equal to 0. Similarly we get $w_2 = 0$, and therefore $u = 0$, q. e. d.

COROLLARY. *We have*

$$\dim (\mathfrak{D}/\mathfrak{D}_0) = 2\tau, \tag{3.36}$$

τ being the excess index of L.

4. Boundary conditions. First we shall study the properties of the skew-symmetric bilinear functionals $[uv](a)$, $[uv](b)$ defined by

$$[uv](a) = \lim_{x \to a} [uv](x), \qquad [uv](b) = \lim_{x \to b} [uv](x), \quad (u, v \varepsilon \mathfrak{D}). \tag{4.1}$$

We start with the important formula:

$$(Tu, v) - (u, Tv) = [u\bar{v}](b) - [u\bar{v}](a), \tag{4.2}$$

deduced immediately from Green's formula. Equation (4.2) shows that the necessary and sufficient condition for u to be contained in $\mathfrak{D}_0$ is

$$[u\bar{v}](b) - [u\bar{v}](a) = 0, \qquad \text{for all } v \varepsilon \mathfrak{D}. \tag{4.3}$$

On the other hand, an arbitrary function $v \varepsilon \mathfrak{D}$ can be decomposed as

$$v = v_1 + v_2, \qquad (v_1, v_2 \varepsilon \mathfrak{D}), \tag{4.4}$$

so that

$$\begin{cases} v_2(x) = 0, & v_1(x) = v(x), \qquad \text{for } a < x \leqq c, \\ v_1(x) = 0, & v_2(x) = v(x), \qquad \text{for } d \leqq x < b, \end{cases}$$

c, d being fixed points such that $a < c < d < b$. For arbitrary $u \varepsilon \mathfrak{D}_0$, we get therefore, by (4. 1) and (4. 3), $[u\bar{v}](b) = [u\bar{v}_2](b) = [u\bar{v}_2](b) - [u\bar{v}_2](a) = 0$, and similarly $[u\bar{v}](a) = 0$. Thus we see, using the invariance of $\mathfrak{D}$ under the conjugation $u \rightarrow \bar{u}$, that *an element u of $\mathfrak{D}$ lies in $\mathfrak{D}_0$ if and only if*

$$(4.5) \qquad [uv](a) = 0, \qquad [uv](b) = 0, \qquad \text{for all } v \varepsilon \mathfrak{D}.$$

Denote by $\mathfrak{D}_a$ (or $\mathfrak{D}_b$) the subspace of $\mathfrak{D}$ consisting of all u satisfying the boundary conditions $[uv](a) = 0$ (or $[uv](b) = 0$) for all $v \varepsilon \mathfrak{D}$. Then we infer by (4. 5)

$$(4.6) \qquad \mathfrak{D}_a \cap \mathfrak{D}_b = \mathfrak{D}_0,$$

while, since v_1, v_2 in (4. 4) are contained in $\mathfrak{D}_b$, $\mathfrak{D}_a$, respectively, we have

$$(4.7) \qquad \mathfrak{D}_a + \mathfrak{D}_b = \mathfrak{D}.$$

Using (3. 36), we conclude from (4. 6) and (4. 7)

$$(4.8) \qquad \dim(\mathfrak{D}/\mathfrak{D}_a) + \dim(\mathfrak{D}/\mathfrak{D}_b) = 2\tau.$$

Moreover *we have*

$$(4.9) \qquad \dim(\mathfrak{D}/\mathfrak{D}_a) = 2\tau_a, \qquad \dim(\mathfrak{D}/\mathfrak{D}_b) = 2\tau_b.$$

To prove this, we observe the operator L in the interval $(a, c]$, $a < c < b$, instead of (a, b), as in the proof of Theorem 3. 2. Then we have $\tau_c = \nu$, and therefore

$$(4.10) \qquad \tau = \tau_a + \tau_c = \tau_a + \nu.$$

On the other hand, we have

$$\mathfrak{D}_c = \{u; u \varepsilon \mathfrak{D}, u(c) = u'(c) = \cdots = u^{(n-1)}(c) = 0\},$$

and consequently $\dim(\mathfrak{D}/\mathfrak{D}_c) = 2\nu$. Combined with (4. 8) and (4. 10), this proves the first formula in (4. 9). The second formula can be proved similarly.

Evidently $[uv](a)$, $[uv](b)$ *can be considered as non-singular bilinear functionals defined respectively in the residual spaces $\mathfrak{D}/\mathfrak{D}_a$, $\mathfrak{D}/\mathfrak{D}_b$.* From (4. 1) it follows that the functionals $[uv](a)$, $[uv](b)$ *are real* in the sense that

$$(4.11) \qquad [\bar{u}\bar{v}](a) = \overline{[uv](a)}, \qquad [\bar{u}\bar{v}](b) = \overline{[uv](b)}.$$

Combined with the invariance of $\mathfrak{D}$ under the conjugation $u \rightarrow \bar{u}$, this yields furthermore

$$(4.12) \qquad \bar{\mathfrak{D}}_a = \mathfrak{D}_a, \qquad \bar{\mathfrak{D}}_b = \mathfrak{D}_b, \qquad \bar{\mathfrak{D}}_0 = \mathfrak{D}_0.$$

By a boundary condition at a, we shall mean a condition for $u \varepsilon \mathfrak{D}$ *of the form* $[\phi u](a) = 0$, *where* ϕ *means a fixed function belonging to* $\mathfrak{D}$. The boundary condition will be called *real,* if ϕ is a real-valued function. A finite number of boundary conditions

$$[\phi_1 u](a) = 0, [\phi_2 u](a) = 0, \cdots, [\phi_m u](a) = 0$$

will be called *linearly independent,* if every linear combination $\Sigma \gamma^j [\phi_j u](a)$ does not vanish identically in u except for the case: $\gamma^1 = \gamma^2 = \cdots = \gamma^m = 0$. *The linear independence of the boundary conditions* $[\phi_1 u](a) = 0, \cdots, [\phi_m u](a) = 0$ *is therefore equivalent to the linear independence of the functions* $\phi_1, \phi_2, \cdots, \phi_m \bmod \mathfrak{D}_a$. Boundary conditions at b, their reality and their linear independence are to be defined similarly.

DEFINITION 4.1. *A system of m linearly independent boundary conditions* $[\phi_j u](a) = 0$ *(or* $[\phi_j u](b) = 0$*)* $(j = 1, 2, \cdots, m)$ *will be called self-adjoint, if* ϕ_j *satisfy* $[\phi_j \phi_k](a) = 0$ *(or* $[\phi_j \phi_k](b) = 0$*)* $(j, k = 1, 2, \cdots, m)$ *and* $m = \tau_a$ *(or* $m = \tau_b$*).*

Our main purpose is to investigate the "eigenvalue problem" for the differential equation $L[u] = \lambda u$ *under the self-adjoint systems of real boundary conditions given at the both ends of the interval a and b.* Let

$$[\phi_{aj} u](a) = 0, \qquad (j = 1, 2, \cdots, \tau_a), \tag{4.13$_a$}$$

$$[\phi_{bj} u](b) = 0, \qquad (j = 1, 2, \cdots, \tau_b), \tag{4.13$_b$}$$

be the given self-adjoint systems of the linearly independent real boundary conditions. Then we denote by $\mathfrak{D}_\phi$ *the subspace of* $\mathfrak{D}$ *consisting of all u satisfying the conditions* $(4.13)_a$, $(4.13)_b$ *and by H the operator with the domain* $\mathfrak{D}_\phi$ *mapping every* $u \varepsilon \mathfrak{D}_\phi$ *in* $L[u]$, *i. e. we put* $Hu = L[u]$, for $u \varepsilon \mathfrak{D}_\phi$. Obviously we have $\mathfrak{D}_0 \subseteqq \mathfrak{D}_\phi \subseteqq \mathfrak{D}$ and therefore

$$T_0 \subseteqq H \subseteqq T. \tag{4.14}$$

Now we shall show that *H is self-adjoint.* For that purpose, we denote for a moment the adjoint operator of H by H^* and the domain of H^* by $\mathfrak{D}_\phi^*$. Then we get from (4.14)

$$T_0 \subseteqq H^* \subseteqq T. \tag{4.15}$$

Combined with (4.2), this shows that $\mathfrak{D}_\phi^*$ consists of all $v \varepsilon \mathfrak{D}$ satisfying

$$[u\bar{v}](b) - [u\bar{v}](a) = 0, \qquad \text{for all } u \varepsilon \mathfrak{D}_\phi. \tag{4.16}$$

Since the bilinear form $[uv](a)$ is non-singular on $\mathfrak{D}/\mathfrak{D}_a$ and dim $(\mathfrak{D}/\mathfrak{D}_a)$

$=2\tau_a$, the condition $(4.13)_a$ determines a linear subspace of $\mathfrak{D}$ which is τ_a-dimensional over $\mathfrak{D}_a$, while, since the system of conditions $(4.13)_a$ is assumed to be self-adjoint, τ_a functions ϕ_{aj} $(j=1,2,\cdots,\tau_a)$ which are linearly independent mod $\mathfrak{D}_a$ satisfy the condition $(4.13)_a$. Hence *the condition* $(4.13)_a$ *is satisfied if and only if* u *can be represented as*

$$u \equiv \sum_j \gamma^j \phi_{aj} \pmod{\mathfrak{D}_a}.$$

Thus we see that *an arbitrary element* $u \varepsilon \mathfrak{D}$ *satisfies the boundary conditions* $(4.13)_a$ *and* $(4.13)_b$ *if and only if* u *can be represented as*

$$(4.17) \qquad u \equiv \sum_j \gamma^j \phi_{aj} \pmod{\mathfrak{D}_a}, \qquad \equiv \sum_j \delta^j \phi_{bj} \pmod{\mathfrak{D}_b}.$$

Moreover, for arbitrary constants $\gamma^1, \gamma^2, \cdots, \delta^1, \delta^2, \cdots$, *we can readily construct a function* $u \varepsilon \mathfrak{D}$ *satisfying* (4.17). Inserting (4.17) in (4.16) and using (4.11), we infer therefore that v belongs to $\mathfrak{D}_\phi{}^*$ if and only if v satisfies the condition

$$\sum_j \bar{\delta}^j [\phi_{bj} v](b) - \sum_j \bar{\gamma}^j [\phi_{aj} v](a) = 0$$

for arbitrary constants $\gamma^1, \gamma^2, \cdots, \delta^1, \delta^2, \cdots$. Hence $\mathfrak{D}_\phi{}^*$ coincides with $\mathfrak{D}_\phi$. Consequently, comparing (4.15) with (4.14), we see that H^* coincides with H. Thus the operator H is self-adjoint.

DEFINITION 4.2. *The operator* H *defined above will be called the self-adjoint realization of the formal differential operator* L *under the system of boundary conditions* $(4.13)_a$ *and* $(4.13)_b$.

5. Spectral theorem. Let H be a fixed self-adjoint realization of the formal differential operator L under the self-adjoint systems of the real boundary conditions

$$(5.1)_a \qquad [\phi_{aj} u](a) = 0, \qquad (j=1,2,\cdots,\tau_a),$$

$$(5.1)_b \qquad [\phi_{bj} u](b) = 0, \qquad (j=1,2,\cdots,\tau_b),$$

in the sense of Definition 4.2. ϕ_{aj} $(j=1,\cdots,\tau_a)$ are therefore assumed to be real functions belonging to $\mathfrak{D}$ which are linearly independent mod $\mathfrak{D}_a$ and satisfy

$$(5.2)_a \qquad [\phi_{aj} \phi_{ak}](a) = 0, \qquad (j,k=1,2,\cdots,\tau_a);$$

similarly ϕ_{bj} $(j=1,\cdots,\tau_b)$ are real functions belonging to $\mathfrak{D}$ which are linearly independent mod $\mathfrak{D}_b$ and satisfy

$$(5.2)_b \qquad [\phi_{bj} \phi_{bk}](b) = 0, \qquad (j,k=1,2,\cdots,\tau_b).$$

DEFINITION 5.1. *By the characteristic planes with respect to the operator H, we shall mean the linear spaces $\mathfrak{p}_a(l)$, $\mathfrak{p}_b(l)$ depending on the parameter l defined respectively by*

$$(5.3)_a \qquad \mathfrak{p}_a(l) = \{(f); f \in \mathfrak{m}_a(l), [\phi_{aj} w(l,f)](a) = 0 \quad (1 \leqq j \leqq \tau_a)\},$$

$$(5.3)_b \qquad \mathfrak{p}_b(l) = \{(f); f \in \mathfrak{m}_b(l), [\phi_{bj} w(l,f)](b) = 0 \quad (1 \leqq j \leqq \tau_b)\},$$

provided $\Im l \neq 0$.

It is to be noted here that the function $w(l,f) = w(x,l,f)$ is not necessarily contained in $\mathfrak{D}$. But, if $f \in \mathfrak{m}_a(l)$, for example, then there exists the limit $[\phi_{aj} w(l,f)](a) = \lim [\phi_{aj} w(l,f)](x)$, as one readily infers from Green's formula; thus Definition 5.1 is legitimate. By a similar reason we infer the existence of the limits

$$[w(l,f)\bar{w}(l,g)](a) = \lim_{x \to a} [w(l,f)\bar{w}(l,g)](x), \qquad (f, g \in \mathfrak{m}_a(l)),$$

$$[w(l,f)\bar{w}(l,g)](b) = \lim_{x \to b} [w(l,f)\bar{w}(l,g)](x), \qquad (f, g \in \mathfrak{m}_b(l)).$$

Incidentally, we conclude, from (2.6), (2.7) and the reality of the boundary conditions $(5.1)_a$, $(5.1)_b$, the relations

$$(5.4) \qquad \mathfrak{p}_a(\bar{l}) = \bar{\mathfrak{p}}_a(l), \qquad \mathfrak{p}_b(\bar{l}) = \bar{\mathfrak{p}}_b(l).$$

THEOREM 5.1. *The characteristic planes $\mathfrak{p}_a(l)$, $\mathfrak{p}_b(l)$ are $(\nu - 1)$-dimensional null planes with respect to l and contained respectively in the sets $\mathfrak{k}_a(l)$, $\mathfrak{k}_b(l)$. Moreover we have*

$$(5.5)_a \qquad [w(l,f)\bar{w}(l,g)](x)/2i\Im l = \int_a^x w(l,f)\bar{w}(l,g)\,dx, \quad (f, g \in \mathfrak{p}_a(l)),$$

$$(5.5)_b \qquad [w(l,f)\bar{w}(l,g)](x)/2i\Im l = -\int_x^b w(l,f)\bar{w}(l,g)\,dx, \quad (f, g \in \mathfrak{p}_b(l)).$$

Proof. We shall prove the theorem with respect to $\mathfrak{p}_a(l)$. For that purpose, we observe the operator L in the interval $(a, c]$, $a < c < b$, instead of (a, b), as in the proof of Theorem 3.2. The meaning of the notations $\mathfrak{H}$, $\mathfrak{D}$, etc. is to be changed correspondingly, so that every $w(x,l,f)$, $f \in \mathfrak{m}_a(l)$, becomes a function contained in $\mathfrak{D}$. Now, as was proved in Section **4** above, every function $u \in \mathfrak{D}$ satisfying $(5.1)_a$ can be represented as $u \equiv \Sigma_j \gamma^j \phi_{aj}$ (mod $\mathfrak{D}_a$). Hence, if $f \in \mathfrak{p}_a(l)$, we have

$$(5.6) \qquad w(l,f) \equiv \sum_j \gamma^j \phi_{aj} \pmod{\mathfrak{D}_a}, \qquad (f \in \mathfrak{p}_a(l)),$$

while, by hypothesis, ϕ_{aj} $(j = 1, \cdots, \tau_a)$ satisfy $(5.2)_a$. Consequently we get $[w(l,f)w(l,g)] = 0$ for $f, g \in \mathfrak{p}_a(l)$, proving that $\mathfrak{p}_a(l)$ is a null plane

with respect to l. Again we conclude from (5. 6), using the reality of ϕ_{aj} and (4. 12), $[w(l,f)\bar{w}(l,g)](a)=0$ for $f,g\,\varepsilon\,\mathfrak{p}_a(l)$. Combined with Green's formula, this yields immediately $(5.5)_a$. Especially for $f=g$, $(5.5)_a$ becomes

$$h(f;x,l)=\int_a^x |w(x,l,f)|^2\,dx, \qquad (f\,\varepsilon\,\mathfrak{p}_a(l)),$$

proving that $h(f;x,l)$ is positive for every f $(\neq 0)$ lying in $\mathfrak{p}_a(l)$. Hence $\mathfrak{p}_a(l)$ is contained in all $\mathfrak{P}^+(x,l)$, $a<x<b$, and therefore $\mathfrak{p}_a(l)\subset\mathfrak{k}_a(l)=\bigcap_x\mathfrak{P}^+(x,l)$. Finally, since $\mathfrak{p}_a(l)$ is the linear subspace of $\mathfrak{m}_a(l)$ determined by τ_a linear equations $[\phi_{aj}w(l,f)](a)=0$ $(j=1,\cdots,\tau_a)$, we have $\dim\mathfrak{p}_a(l)\geqq\dim\mathfrak{m}_a(l)-\tau_a=\nu-1$. On the other hand, $\mathfrak{k}_a(l)$ has no common point with $\mathfrak{k}_c(l)$ and, as Theorem 2. 1 shows, $\mathfrak{k}_c(l)$ contains (at least) one $(\nu-1)$-dimensional null plane $\mathfrak{p}$ with respect to l. Hence $\mathfrak{p}_a(l)$ has no common point with $\mathfrak{p}$ and therefore

$$\dim\mathfrak{p}_a(l)\leqq\dim\mathfrak{P}-\dim\mathfrak{p}-1=\nu-1.$$

Thus we see that $\dim\mathfrak{p}_a(l)$ is equal to $\nu-1$.

THEOREM 5. 2. *As functions of the parameter l, the characteristic planes $\mathfrak{p}_a(l)$, $\mathfrak{p}_b(l)$ are holomorphic, provided $\Im l\neq 0$, in the sense that, for every l_0 with $\Im l_0\neq 0$, the suitably normalized*[20] *Plücker's coordinates $\mathfrak{p}_a^{ij\ldots k}(l)$, $\mathfrak{p}_b^{ij\ldots k}(l)$ of $\mathfrak{p}_a(l)$, $\mathfrak{p}_b(l)$ are holomorphic with respect to l in some neighborhoods of l_0.*

Proof. We shall prove the theorem with respect to $\mathfrak{p}_a(l)$. For that purpose, we observe the operator L in the interval $(a,c]$, $a<c<b$, as above. Then we have $\dim\mathfrak{E}(l)=\tau_a+\nu$.

Now let D be an arbitrary compact domain in the half-plane $\Im l>0$ (or $\Im l<0$). Then, by virtue of Lemma 3. 1, the base $r_\sigma(x,l)$ $(\sigma=1,2,\cdots,\tau_a+\nu)$ of the space $\mathfrak{E}(l)$ can be chosen so that $r_\sigma(x,l)$ are holomorphic in D with respect to l in the sense of the norm: $\|\ \ \|$ and that $d^m r_\sigma(x,l)/dx^m$ $(m=0,\cdots,n-1)$ are holomorphic in D with respect to l. Putting $r_\sigma(x,l)=w(x,l,f_\sigma(l))$, we obtain the points $f_\sigma(l)$ $(\sigma=1,\cdots,\tau_a+\nu)$ generating the space $\mathfrak{m}_a(l)$. Then an arbitrary point f in $\mathfrak{m}_a(l)$ can be represented as $f=\Sigma\,\eta^\sigma f_\sigma(l)$, so that have $w(x,l,f)=\Sigma\,\eta^\sigma r_\sigma(x,l)$. Using these η^σ $(\sigma=1,2,\cdots,\tau_a+\nu)$ as coordinates of points f in $\mathfrak{m}_a(l)$, we infer from $(5.3)_a$ that the subspace $\mathfrak{p}_a(l)$ of $\mathfrak{m}_a(l)$ is determined by the linear equations

$$(5.7)\qquad \Sigma\,\eta^\sigma[\phi_{aj}r_\sigma(l)](a)=0, \qquad (j=1,2,\cdots,\tau_a).$$

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[20] See footnote 14.

These τ_a equations are linearly independent, since $\dim \mathfrak{p}_a(l) = \dim \mathfrak{m}_a(l) - \tau_a$; while the coefficients $[\phi_{aj} r_\sigma(l)](a)$ are holomorphic in D with respect to l, as one readily infers from

$$[\phi_{aj} r_\sigma(l)](a) = [\phi_{aj} r_\sigma(l)](c) - \int_a^c (L[\phi_{aj}] - l\phi_{aj}) r_\sigma(l)\, dx.$$

Consequently the ν linearly independent solutions $\eta_\alpha{}^\sigma(l)$ $(\alpha = 1, \cdots, \nu)$ of the equations (5.7) can be chosen so that $\eta_\alpha{}^\sigma(l)$ are holomorphic in D with respect to l. On the other hand, $f_\sigma(l)$ $(\sigma = 1, 2, \cdots, \tau_a + \nu)$ are also holomorphic in D with respect to l, as was shown in the proof of Theorem 3.3. Hence putting $g_\alpha(l) = \Sigma_\sigma \eta_\alpha{}^\sigma(l) f_\sigma(l)$, we obtain ν points $g_1(l), \cdots, g_\nu(l)$ depending on l holomorphically in D which generate the characteristic plane $\mathfrak{p}_a(l)$. This proves that $\mathfrak{p}_a(l)$ depends on l holomorphically, provided $\Im l \neq 0$.

After these preparatory considerations, we shall now proceed to the investigation of the spectra of H. First we introduce

Definition 5.2. *By the characteristic matrix of the operator H with respect to the regular system of fundamental solutions $s_j(x, l)$ $(j = 1, 2, \cdots, n)$ we shall mean the matrix $M^{jk}(l)$ defined by*

$$M^{jk}(l) = M^{jk}(l, \mathfrak{p}_a(l), \mathfrak{p}_b(l)), \qquad (\Im l \neq 0), \tag{5.8}$$

$\mathfrak{p}_a(l)$, $\mathfrak{p}_b(l)$ being the characteristic planes with respect to H (see (3.6)).

As (3.7) shows, *the characteristic matrix $M^{jk}(l)$ is determined by the* relation

$$\sum_k \sum_m M^{mj}(l)[s_m(l) s_k(l)] \cdot f^k = \begin{cases} f^j, & (\text{for } f \in \mathfrak{p}_a(l)), \\ 0, & (\text{for } f \in \mathfrak{p}_b(l)). \end{cases} \tag{5.9}$$

Combined with (5.4) and (2.3), this yields

$$M^{jk}(\bar{l}) = \bar{M}^{jk}(l). \tag{5.10}$$

Again, using (5.9), we conclude from Theorem 5.2 that *the characteristic matrix $M^{jk}(l)$ depends on l holomorphically, provided $\Im l \neq 0$.*

Definition 5.3. *By Green's function for the operator H will be meant the function $G(x, y; l)$ defined as*

$$G(x, y; l) = G(x, y; l, \mathfrak{p}_a(l), \mathfrak{p}_b(l)), \qquad (\Im l \neq 0), \tag{5.11}$$

$\mathfrak{p}_a(l)$, $\mathfrak{p}_b(l)$ being characteristic planes with respect to H. The integral operator with the kernel $G(x, y; l)$ will be denoted by $G(l)$, i.e., *$G(l) = G(l, \mathfrak{p}_a(l), \mathfrak{p}_b(l))$ (see Definition 3.2).*

By using the characteristic matrix, Green's function for H can be written in the form

$$G(x,y;l) = G(y,x;l) = \sum_{j,k} M^{jk}(l) s_j(x,l) s_k(y,l), \qquad (x \geqq y), \tag{5.12}$$

as (3.9) shows. From (5.10) and (2.3) follows therefore

$$G(x,y,\bar{l}) = \bar{G}(x,y,l). \tag{5.13}$$

Now we shall prove

THEOREM 5.3. *The integral operator $G(l)$ coincides with the resolvent $(H-l)^{-1}$ of the operator H, provided $\Im l \neq 0$.*

Proof. Denoting by $\mathfrak{D}_H$ the domain of H, we first prove that, for arbitrary $v \,\varepsilon\, \mathfrak{H}$, $G(l)v$ belongs to $\mathfrak{D}_H$. For that purpose, decompose v into two parts:

$$v = v_0 + \bar{w}, \qquad (v_0 \,\varepsilon\, \mathfrak{H} - \bar{\mathfrak{E}}(l), w \,\varepsilon\, \mathfrak{E}(l)).$$

Then, by Theorem 3.5, $G(l)v_0$ belongs to $\mathfrak{D}_0 \subset \mathfrak{D}_H$; hence, for the present purpose, it is sufficient to show that $G(l)\bar{w}$ is contained in $\mathfrak{D}_H$. Now we have, by (3.11),

$$G(l)\bar{w}(x) = \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l)\{w_\beta(x) \int_a^x w_\alpha \bar{w}\, dx + w_\alpha(x) \int_x^b w_\beta \bar{w}\, dx\},$$

while, using Green's formula, we get

$$2i\Im l \cdot \int_a^x w_\alpha \bar{w}\, dx = [w_\alpha \bar{w}](x) - [w_\alpha \bar{w}](a),$$

$$2i\Im l \cdot \int_x^b w_\beta \bar{w}\, dx = [w_\beta \bar{w}](b) - [w_\beta \bar{w}](x).$$

By virtue of the identity (3.18), we conclude therefore

$$G(l)\bar{w}(x) = (i/2\Im l)\bar{w}(x) + (i/2\Im l) \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l)\{[w_\alpha \bar{w}](a) \cdot w_\beta(x) - [w_\beta \bar{w}](b) \cdot w_\alpha(x)\}. \tag{5.14}$$

Let ψ_j $(j = 1, \cdots, \tau)$ be real functions belonging to $\mathfrak{D}$ such that

$$\psi_j(x) = \phi_{aj}(x) \text{ (for } a < x \leqq c), = 0 \text{ (for } d \leqq x < b), \qquad (j = 1, \cdots, \tau_a),$$

$$\psi_{\tau_a+j}(x) = 0 \text{ (for } a \leqq x < c), = \phi_{bj}(x) \text{ (for } d \leqq x < b), \ (j = 1, \cdots, \tau_b),$$

c, d being fixed points such that $a < c < d < b$ (such functions can be readily constructed). Then ψ_j $(j = 1, 2, \cdots, \tau)$ are linearly independent mod $\mathfrak{D}_0$ and satisfy the conditions

$$[\psi_j \psi_k](b) - [\psi_j \psi_k](a) = 0, \qquad (j, k = 1, 2, \cdots, \tau). \tag{5.15}$$

By virtue of Theorem 3. 6, ψ_j can be decomposed uniquely as $\psi_j \equiv t_j + \bar{r}_j$ $(\mathrm{mod}\, \mathfrak{D}_0)$ $(t_j, r_j \,\varepsilon\, \mathfrak{E}(l))$, while, since ψ_j are real and $\mathfrak{D}_0 = \bar{\mathfrak{D}}_0$, from the uniqueness of the decomposition it follows that t_j must coincide with r_j. Thus we get

$$\psi_j \equiv r_j + \bar{r}_j \ (\mathrm{mod}\, \mathfrak{D}_0), \qquad (r_j \,\varepsilon\, \mathfrak{E}(l)). \tag{5.16}$$

Inserting this in (5. 15), we obtain $\Re\{[r_j\bar{r}_k](b) - [r_j\bar{r}_k](a)\} = 0$, while from Green's formula follows $[r_j\bar{r}_k](b) - [r_j\bar{r}_k](a) = 2i\Im l \cdot (r_j, r_k)$. Hence the inner products (r_j, r_k) $(j, k = 1, 2, \cdots, \tau)$ are all real. From this we conclude, using (5. 16), that the functions $r_1, r_2, \cdots, r_\tau$ are linearly independent. Indeed, if $r_1, r_2, \cdots, r_\tau$ were linearly dependent, then there would exist *non-trivial* linear relations $\Sigma_j \eta^j r_j = 0$ with *real* coefficients $\eta^1, \eta^2, \cdots, \eta^\tau$. Whence one would get

$$\Sigma \eta^j \psi_j \equiv \Sigma \eta^j r_j + \Sigma \eta^j \bar{r}_j = 0 \ (\mathrm{mod}\, \mathfrak{D}_0),$$

contradicting with the linear independence of $\psi_1, \psi_2, \cdots, \psi_\tau \bmod \mathfrak{D}_0$. Thus the functions $r_1, r_2, \cdots, r_\tau$ are linearly independent and therefore constitute a base of the space $\mathfrak{E}(l)$. Consequently the function $\bar{w}$ appearing in (5. 14) can be represented as $\bar{w} = \Sigma \gamma^j \bar{r}_j$. Put now $r = \Sigma \gamma^j r_j$. Then we get from (5. 16)

$$r + \bar{w} \equiv \Sigma \gamma^j \psi_j \ (\mathrm{mod}\, \mathfrak{D}_0), \qquad (r \,\varepsilon\, \mathfrak{E}(l)). \tag{5.17}$$

The functions w_α $(\alpha = 1, \cdots, \nu)$ satisfy the boundary conditions $(5.1)_a$. Hence we have $[\psi_j w_\alpha](a) = 0$ $(j = 1, \cdots, \tau;\ \alpha = 1, 2, \cdots, \nu)$. From (5. 17) follows therefore $[w_\alpha \bar{w}](a) = -[w_\alpha r]$, $(\alpha = 1, 2, \cdots, \nu)$ (remember that $[w_\alpha r](x)$ does not depend on x); similarly we get

$$[w_\beta \bar{w}](b) = -[w_\beta r], \qquad (\beta = \nu + 1, \cdots, n).$$

Inserting these identities in (5. 14) and using (3. 18), we obtain

$$G(l)\bar{w} = (i/2\Im l)\{\bar{w} + r\} \equiv (i/2\Im l)\Sigma \gamma^j \psi_j \ (\mathrm{mod}\, \mathfrak{D}_0),$$

while it is obvious that ψ_j $(j = 1, \cdots, \tau)$ satisfy the boundary conditions $(5.1)_a$ and $(5.1)_b$. Hence $G(l)\bar{w}$ satisfies also $(5.1)_a$ and $(5.1)_b$, and therefore belongs to $\mathfrak{D}_H$. Thus we conclude that $G(l)v$ belongs to $\mathfrak{D}_H$ for every $v \,\varepsilon\, \mathfrak{H}$. This shows, combined with (3. 22), that $(H - l)G(l) = 1$.

To prove the transposed relation

$$G(l)(H - l)u = u, \qquad \text{for all } u \,\varepsilon\, \mathfrak{D}_H, \tag{5.18}$$

we represent, using Theorem 3. 4, every $u \,\varepsilon\, \mathfrak{D}_H$ in the form $u = G(l)v + w$, $w \,\varepsilon\, \mathfrak{E}(l)$, where $v = (T - l)u$. Then, since $G(l)v$ belongs to $\mathfrak{D}_H$, w is also

contained in $\mathfrak{D}_H$ and therefore $Hw = L[w] = lw$ $(\Im l \neq 0)$, whereas the self-adjoint operator H has no complex eigenvalue. Hence w must vanish, so that we obtain $u = G(l)v = G(l)(T - l)u = G(l)(H - l)u$, proving (5.18), q. e. d.

Now that Theorem 5.3 has been established, our theory proceeds along the line completely parallel to the case of the differential operator of the second order.[21] The main results can be stated as follows:

THEOREM 5.4. (SPECTRAL THEOREM). *Let*

$$H = \int_{-\infty}^{+\infty} \lambda dE(\lambda)$$

be the spectral representation of the self-adjoint operator H and $M^{jk}(l)$ the characteristic matrix of H with respect to the regular system of fundamental solutions $s_j(x, l)$ $(j = 1, 2, \cdots, n)$. Then for every real number λ, there exists the limit

$$\rho^{jk}(\lambda) = \lim_{\delta \to +0} \lim_{\epsilon \to +0} (1/\pi) \int_{\delta}^{\lambda+\delta} \Im M^{jk}(\lambda + i\epsilon) d\lambda. \tag{5.19}$$

The matrix $\rho(\lambda) = (\rho^{jk}(\lambda))$ thus defined is real and symmetric. As a function of λ, $\rho(\lambda)$ is continuous on the right and monotone non-decreasing in the sense that, for $\mu < \lambda$, the symmetric matrix $\rho(\lambda) - \rho(\mu)$ is positive semi-definite. Put $E(\Delta) = E(\lambda) - E(\mu)$ for every finite interval $\Delta = (\mu, \lambda]$. Then, for every $u \in \mathfrak{H}$, $E(\Delta)u(x)$ can be represented as

$$E(\Delta)u(x) = \int_a^b u(y)dy \int_{\Delta} \sum_{j,k} s_j(x, \lambda) s_k(y, \lambda) d\rho^{jk}(\lambda), \tag{5.20}$$

where

$$\int_a^b dy \left| \int_{\Delta} \sum_{j,k} s_j(x, \lambda) s_k(y, \lambda) d\rho^{jk}(\lambda) \right|^2 < +\infty,$$

and the integral with respect to y in (5.20) converges absolutely. Thus $E(\Delta)$ is the integral operator with the symmetric kernel

$$E(x, y; \Delta) = \int_{\Delta} \sum_{j,k} s_j(x, \lambda) s_k(y, \lambda) d\rho^{jk}(\lambda)$$

of Carleman type. $u(x)$ is represented as

$$u(x) = \lim_{\lambda \to +\infty, \mu \to -\infty} \int_a^b u(y)dy \int_{\mu}^{\lambda} \sum_{j,k} s_j(x, \lambda) s_k(y, \lambda) d\rho^{jk}(\lambda), \tag{5.21}$$

[21] Kodaira, [3].

where the limit converges in the mean. The formula (5.19) *can be rewritten as*

$$(5.22)\qquad \rho^{jk}(\lambda) = \lim_{\delta\to+0} \lim_{\epsilon\to+0} (1/\pi)\Im\{ \int_{C(\lambda+\delta,\delta,\epsilon)} M^{jk}(l)\,dl\},$$

where $C(\lambda+\delta,\delta,\epsilon)$ *means the oriented polygonal line whose vertices, in order, are* $\delta+i\epsilon$, $\delta+i\alpha$, $\lambda+\delta+i\alpha$, $\lambda+\delta+i\epsilon$, *the real number* α *being subject to the inequality* $\alpha>\epsilon$.[22]

Proof. First we shall treat the special case that the fundamental solutions $s_j(x,l)$ $(j=1,2,\cdots,n)$ constitute the canonical system obtained by solving the differential equation: $L[s]=ls$ under the *real* boundary conditions

$$(5.23)\qquad [d^m s_j(x,l)/dx^m]_{x=c} = e_j^{(m)}, \quad (m=0,\cdots,n-1;\, j=1,\cdots,n)$$

at the fixed point c, $a<c<b$, where $e_j=(e_j^{(0)},e_j^{(1)},\cdots,e_j^{(n-1)})$ are constant real vectors satisfying

$$(5.24)\qquad [e_je_k](c)=\epsilon_{jk}$$

($s_j(x,l)$ mean therefore the solutions $s_j^0(x,l)$ mentioned in Section **2**). We follow the method of H. Weyl.[23] For arbitrary $u.\varepsilon H$, we put

$$u(x,\lambda)=[E(\lambda)-E(0)]u(x),$$
$$u(x,\Delta)=u(x,\lambda)-u(x,\mu)=E(\Delta)u(x), \qquad (\Delta=(\mu,\lambda]);$$

considering $u(x,\Delta)$ as an element of $\mathfrak{H}$, we use the abbreviation $u(\Delta)$ for $u(x,\Delta)$. Obviously $u(\Delta)$ belongs to the domain of H. Putting

$$(5.25)\qquad v(\Delta)=(H-l)u(\Delta)=\int_\Delta(\lambda-l)\,dE(\lambda)u,$$

we have therefore, by Theorem 5.3,

$$(5.26)\qquad u(\Delta)=G(l)v(\Delta).$$

Put, for simiplicity's sake, $G^{(m)}(x,y;l)=\partial^m G(x,y;l)/\partial x^m$, $(y\neq x)$. Then we get from (5.26)

$$(5.27)\qquad \partial^m u(x,\Delta)/\partial x^m=\int_a^b G^{(m)}(x,y;l)v(y,\Delta)\,dy, \qquad (m=0,\cdots,n-1),$$

as (3.17) shows. The explicit form of $G^{(m)}(x,y;l)$ is given by

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[22] In the case of second-order differential operators, this theorem is reduced to the Weyl-Stone-Titchmarsh results. See Weyl, [11], [12], Stone, [7], Theorem 10.22, Titchmarsh, [9], Chap. III. Cf. also Kodaira, [3].

[23] Weyl, [11].

$$(5.28)\quad G^{(m)}(x,y;l)=\begin{cases}\sum_{\alpha=1}^{\nu}\sum_{\beta=\nu+1}^{n}F_{\alpha\beta}(l)w_\beta{}^{(m)}(x)w_\alpha(y), & (y<x),\\ \sum_{\alpha=1}^{\nu}\sum_{\beta=\nu+1}^{n}F_{\alpha\beta}(l)w_\alpha{}^{(m)}(x)w_\beta(y), & (y>x),\end{cases}$$

whence we see that $G^{(m)}(x,y;l)$ is square summable with respect to y in (a,b), provided $a<x<b$. Thus, *for fixed x, the functions $G^{(m)}(x,y;l)$ of y can be considered as an element of $\mathfrak{H}$; then $G^{(m)}(x,y;l)$ will be denoted by $G^{(m)}(x;l)$.* By using (5.13), the identity (5.27) can be rewritten as

$$u^{(m)}(x,\Delta)=(v(\Delta),G^{(m)}(x;\bar{l})).$$

Inserting (5.25) for $v(\Delta)$, we obtain from this the important formulae

$$(5.29)\quad u^{(m)}(x,\Delta)=(u,\int_\Delta(\lambda-\bar{l})dE(\lambda)G^{(m)}(x;\bar{l})),\quad (m=0,1,\cdots,n-1).$$

Now, using these formulae, we shall prove that the functions $u^{(m)}(x,\lambda)$ $(m=0,1,\cdots,n-1)$ have the following properties:

(#) *As a function of λ, each $u^{(m)}(x,\lambda)$ is continuous on the right, of bounded variation in every finite interval $\Delta=(\mu,\lambda]$ and, for such intervals Δ, the total variation $\int_\Delta|du^{(m)}(x,\lambda)|$ is uniformly bounded with respect to x in every closed interval $[x_1,x_2]$, $a<x_1<x_2<b$.*

The formulae (5.29) can be rewritten as

$$u^{(m)}(x,\Delta)=(u(\Delta),\int_\Delta(\lambda-\bar{l})dE(\lambda)G^{(m)}(x;\bar{l})),$$

yielding immediately

$$(5.30)\quad |u^{(m)}(x,\Delta)|\leqq\|u(\Delta)\|\cdot\{\int_\Delta|\lambda-l|^2d\|E(\lambda)G^{(m)}(x;\bar{l})\|^2\}^{1/2}.$$

Consider now decompositions of Δ into the sum of a finite number of sub-intervals $\Delta_j=(\lambda_{j-1},\lambda_j]$, λ_j being real numbers such that $\lambda_0=\mu<\lambda_1<\cdots<\lambda_h=\lambda$. Then, applying (5.30) to each Δ_j, we get

$$\int_\lambda|du^{(m)}(x,\lambda)|=\sup\sum_j|u^{(m)}(x,\Delta_j)|$$

$$\leqq\|u(\Delta)\|\cdot\{\int_\Delta|\lambda-l|^2d\|E(\lambda)G^{(m)}(x;\bar{l})\|^2\}^{1/2}.$$

Hence, denoting by α the maximum of $|\lambda-l|$ in Δ, we obtain

$$(5.31) \qquad \int_\Delta | du^{(m)}(x,\lambda) | \leqq \alpha \| u(\Delta) \| \cdot \| G^{(m)}(x;l) \|,$$

proving that $u^{(m)}(x,\lambda)$ is a function of bounded variation in Δ, while it is obvious that $u^{(m)}(x,\lambda)$ is continuous on the right (we have naturally presupposed that $E(\lambda)$ is continuous on the right in the sense of the strong topology). Again, as one readily infers by (5.28), $\| G^{(m)}(x;l) \|$ is uniformly bounded with respect to x in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$. Combined with (5.31), this proves that the total variation $\int_\Delta | du^{(m)}(x,\lambda) |$ is uniformly bounded in $[x_1, x_2]$. Thus $u^{(m)}(x,\lambda)$ have the properties (#).

As an element of $\mathfrak{H}$, $u(\Delta)$ satisfies obviously the equation

$$(5.32) \qquad Hu(\Delta) = \int_\Delta \lambda dE(\lambda) u(\Delta).$$

Using this we shall prove that *the function $u(x,\lambda)$ satisfies the integro-differential equation*

$$(5.33) \qquad L[u(x,\lambda)] = \int_{+0}^{\lambda} \lambda du(x,\lambda).$$

For that purpose, let $\lambda_0, \lambda_1, \cdots, \lambda_h$ be real numbers such that $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_h = \lambda$ and put $\Delta_j = (\lambda_{j-1}, \lambda_j]$, $\delta = \max_j | \lambda_j - \lambda_{j-1} |$. Then we have

$$| \sum_j \lambda_j u(x,\Delta_j) - \int_{+0}^{\lambda} \lambda du(x,\lambda) | \leqq \delta \cdot \int_{+0}^{\lambda} | du(x,\lambda) |.$$

This shows that, in the relation

$$(5.34) \qquad \lim_{\delta \to 0} \sum_j \lambda_j u(x,\Delta_j) = \int_{+0}^{\lambda} \lambda du(x,\lambda),$$

the limit converges uniformly with respect to x in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$, since the total variation $\int_{+0}^{\lambda} | du(x,\lambda) |$ is uniformly bounded in $[x_1, x_2]$. On the other hand, applied to $\Delta = (0, \lambda]$, (5.32) means

$$\lim_{\delta \to 0} \int_a^b | L[u(x,\lambda)] - \sum_j \lambda_j u(x,\Delta_j) |^2 dx = 0.$$

Hence we get (5.33).

Incidentally, from (5.34) it follows that $\int_{+0}^{\lambda} \lambda du(x,\lambda)$ is a continuous function of x in $a < x < b$. Combined with (5.33), this shows that, *as a function of x, $u(x,\lambda)$ admits in (a,b) the continuous n-th derivative.*

Now we introduce the functions $u_j(\lambda)$ $(j = 1, 2, \cdots, n)$ of λ associated with $u(x,\lambda)$ by the relations

$$(5.35)\qquad \sum_{j=1}^{n} e_j^{(m)} u^j(\lambda) = u^{(m)}(c,\lambda), \qquad (m = 0, 1, \cdots, n-1),$$

where $e_j^{(m)}$ mean the constants appearing in the boundary conditions (5.23) and therefore $\det(e_j^{(m)}) \neq 0$. Obviously $u^j(\lambda)$ $(j = 1, 2, \cdots, n)$ *are functions of bounded variation in every finite interval* Δ. By using these functions $u_j(\lambda)$, *the function* $u(x,\lambda)$ *can be represented as*

$$(5.36)\qquad u(x,\lambda) = \sum_{j=1}^{n} \int_{+0}^{\lambda} s_j(x,\lambda)\, du^j(\lambda).$$

To prove this we consider the difference

$$t(x,\lambda) = u(x,\lambda) - \sum_{j=1}^{n} \int_{+0}^{\lambda} s_j(x,\lambda)\, du^j(\lambda).$$

As a function of x, $t(x,\lambda)$ admits also continuous derivatives up to the order n and, as a function of λ, $t(x,\lambda)$ is of bounded variation in every finite interval Δ. Furthermore it can be readily verified that $t(x,\lambda)$ satisfies

$$(5.37)\qquad L[t(x,\lambda)] = \int_{+0}^{\lambda} \lambda\, dt(x,\lambda),$$

while, since $s_j^{(m)}(c,\lambda) = e_j^{(m)}$, from (5.35) follows

$$(5.38)\qquad t^{(m)}(c,\lambda) = 0, \qquad (m = 0, 1, \cdots, n-1).$$

Now, by hypothesis, we have

$$\sum_{h+m \leqq n-1} B_{mh}(x) s_j^{(m)}(x,\lambda) s_k^{(h)}(x,\lambda) = [s_j s_k] = \epsilon_{jk},$$

whence we obtain, using (1.6),

$$\sum_{j,k} \epsilon_{jk} s_j^{(m)}(x,\lambda) s_k(x,\lambda) = 0 \ \ (\text{for } 0 \leqq m \leqq n-2), = 1/p_0(x)$$
$$(\text{for } m = n-1).$$

Hence, putting

$$W(x,y) = \sum_{j,k} \epsilon_{jk} s_j(x,0) s_k(y,0),$$

we infer readily that, under the boundary conditions (5.38), the integro-differential equation (5.37) is equivalent to the integral equation

$$t(x,\lambda) = \int_c^x W(x,y)\, dy \int_{+0}^{\lambda} \lambda\, dw(y,\lambda).$$

Using the method of iteration, we readily infer from this that $t(x,\lambda)$ must vanish identically. Thus (5.36) has been proved.

The explicit forms of $u_j(\lambda)$ are obtained from (5.35) and (5.29). Putting

$$(5.39)\qquad \Gamma^j(x;l)=\begin{cases}\sum_k M^{jk}(l)s_k(x,l), & \text{for } x\leqq c,\\ \sum_k M^{kj}(l)s_k(x,l), & \text{for } x>c,\end{cases}$$

we obtain, using (5.12) and (5.23), $G^{(m)}(c,x;l)=\sum_j e_j{}^{(m)}\Gamma^j(x;l)$. From (5.29) follows therefore

$$(5.40)\qquad u^{(m)}(c,\Delta)=\sum_j e_j{}^{(m)}(u,\int_\Delta(\lambda-\bar{l})dE(\lambda)\Gamma^j(\bar{l})).$$

Introduce now the functions $\xi^j(\Delta)=\xi^j(x,\Delta)$ defined by

$$(5.41)\qquad \xi^j(\Delta)=\int_\Delta(\lambda-l)dE(\lambda)\Gamma^j(\bar{l})$$

Then, comparing (5.40) with (5.35), we get the important formulae

$$(5.42)\qquad u^j(\Delta)=(u,\xi^j(\Delta)),\qquad (j=1,2,\cdots,n).$$

As (5.35) shows, $u^j(\Delta)$ can be considered for fixed Δ as a linear functional of $u\,\varepsilon\,\mathfrak{H}$ *not depending on* l. Hence we infer by (5.42) that *the functions* $\xi^j(\Delta)$ $(j=1,2,\cdots,n)$ *are independent of* l, *whereas the definition* (5.41) *of* $\xi^j(\Delta)$ *contains the parameter* l. Obviously $\xi^j(\Delta)$ belongs to $\mathfrak{H}$ and, as (5.41) shows,

$$(5.43)\qquad (\xi^j(\Delta'),\xi^k(\Delta''))=0,\qquad \text{if } \Delta'\cap\Delta'' \text{ is empty.}$$

The inner products $(\xi^j(\Delta),\xi^k(\Delta))$ *are therefore additive functions of intervals* Δ.

To prove the existence of the limit (5.19), we first define the additive functions $\rho^{jk}(\Delta)$ of intervals Δ by

$$(5.44)\qquad \rho^{jk}(\Delta)=(\xi^j(\Delta),\xi^k(\Delta)),$$

and then prove that $\rho^{jk}(\lambda)=\rho^{jk}((0,\lambda])$ are related to $M^{jk}(l)$ by (5.19) (in what follows we assume that the additive functions $\rho^{jk}(\Delta)$, $\rho^{0jk}(\Delta)$ etc. of the intervals Δ are always related to the corresponding functions $\rho^{jk}(\lambda)$, $\rho^{0jk}(\lambda)$, etc. of λ by the equations as follows:

$$\rho^{jk}(\lambda)=\rho^{jk}((0,\lambda]),\qquad \rho^{jk}(\Delta)=\rho^{jk}(\lambda)-\rho^{jk}(\mu)\qquad (\Delta=(\mu,\lambda)).$$

Evidently the matrix $\rho(\Delta)=(\rho^{jk}(\Delta))$ is *hermitian* and *positive semi-definite*; therefore we have

$$(5.45)\qquad \rho^{jj}(\Delta)\geqq 0,$$

$$(5.46)\qquad |\rho^{jk}(\Delta)|^2\leqq\rho^{jj}(\Delta)\rho^{kk}(\Delta).$$

From (5.46) follows that $\rho^{jk}(\lambda)$ *are functions of bounded variation in every finite interval* Δ. Now, inserting (5.41) in (5.44), we get

$$\rho^{jk}(\Delta) = \int_\Delta |\lambda - l|^2 (dE(\lambda)\Gamma^j(\bar{l}), \Gamma^k(\bar{l})),$$

whence we conclude

$$\int_{-\infty}^{+\infty} |\lambda - l|^{-2} d\rho^{jk}(\lambda) = (\Gamma^j(\bar{l}), \Gamma^k(\bar{l})), \tag{5.47}$$

where the integral converges absolutely, since, by (5.46),

$$\{\int |\lambda - l|^{-2} |d\rho^{jk}(\lambda)|\}^2 \leqq \int |\lambda - l|^{-2} d\rho^{jj}(\lambda) \cdot \int |\lambda - l|^{-2} d\rho^{kk}(\lambda)$$
$$\leqq \|\Gamma^j\|^2 \|\Gamma^k\|^2.$$

The right hand side of (5.47) can be readily calculated. Inserting the explicit form (3.6) of the matrix $M^{jk}(l)$ into (5.39), we get

$$\Gamma^j(x; l) = \sum_{\alpha=1}^{\nu} \sum_{\beta=\nu+1}^{n} F_{\alpha\beta}(l) f_\beta{}^j w(x, l, f_\alpha), \qquad (x \leqq c),$$

where f_α lie in $\mathfrak{p}_\alpha(l)$. Using $(5.5)_a$, (5.39), (5.23) and (5.24), we obtain therefore

$$2i\Im l \int_a^c \Gamma^k(x; l)\Gamma^j(x; \bar{l})dx = [\Gamma^k(l)\Gamma^j(\bar{l})](c-0)$$
$$= \sum_h \sum_m M^{kh}(l)M^{jm}(\bar{l}) \cdot [s_h(l)s_m(\bar{l})](c) = \sum_{h,m} M^{kh}(l)M^{jm}(\bar{l})\epsilon_{hm};$$

similarly we have

$$2i\Im l \int_c^b \Gamma^k(x; l)\Gamma^j(x; \bar{l})dx = -\sum_{h,m} M^{hk}(l)M^{mj}(\bar{l})\epsilon_{hm}.$$

Thus we get

$$2i\Im l \cdot (\Gamma^j(\bar{l}), \Gamma^k(\bar{l})) = \sum_{h,m} \{M^{kh}(l)M^{jm}(\bar{l}) - M^{hk}(l)M^{mj}(\bar{l})\}\epsilon_{hm}.$$

Combined with (5.10) and the formula $\Sigma_h[M^{hj}(l) - M^{jh}(l)]\epsilon_{hk} = \delta_k{}^j$, deduced from (3.10), this yields immediately

$$\Im l \cdot (\Gamma^j(\bar{l}), \Gamma^k(\bar{l})) = \Im M^{jk}(l).$$

Hence we get from (5.47) the *important formulae*

$$\int_{-\infty}^{+\infty} \Im l \cdot |\lambda - l|^{-2} d\rho^{jk}(\lambda) = \Im M^{jk}(l), \tag{5.48}$$

where the integral converges absolutely. Now the relation (5.19) can be readily deduced from (5.48). Indeed we have

$$\lim_{\epsilon\to+0}\int_\delta^{\lambda+\delta}\Im M^{jk}(\lambda+i\epsilon)\,d\lambda=\lim_{\epsilon\to+0}\int_\delta^{\lambda+\delta}d\lambda\int_{-\infty}^{+\infty}\epsilon[(\mu-\lambda)^2+\epsilon^2]^{-1}d\rho^{jk}(\mu)$$
$$=\int_{-\infty}^{+\infty}d\rho^{jk}(\mu)\lim_{\epsilon\to+0}[\tan^{-1}((\lambda+\delta-\mu)/\epsilon)-\tan^{-1}((\delta-\mu)/\epsilon)]$$
$$=(\pi/2)[\rho^{jk}(\lambda+\delta)+\rho^{jk}(\lambda+\delta-0)-\rho^{jk}(\delta)-\rho^{jk}(\delta-0)],$$

yielding immediately (5. 19).

As was already proved, the matrix $\rho(\Delta)=(\rho^{jk}(\Delta))$ is Hermitian, while (5. 19) shows that $\rho(\Delta)$ is real; hence $\rho(\Delta)$ must be *symmetric.*

Now, inserting $u=\xi^k(\Delta)$ into (5. 42) and (5. 36), we get readily the expression

$$\xi^k(x,\Delta)=\int_\Delta\sum s_j(x,\lambda)\,d\rho^{kj}(\lambda). \tag{5.49}$$

Using this, we shall prove the formula (5. 20) for $\Delta=(\mu,\lambda]$. For that purpose, let $\lambda_0,\lambda_1,\cdots,\lambda_h$ be real numbers such that $\mu=\lambda_0<\lambda_1<\cdots<\lambda_h=\lambda$, put $\delta=\max_m|\lambda_m-\lambda_{m-1}|$, and consider for fixed x, $a<x<b$, the sum

$$\Xi(y;\delta)=\sum_m\sum_k s_k(x,\lambda_m)\xi^k(y,\Delta_m),\qquad \Delta_m=(\lambda_{m-1},\lambda_m].$$

Choose for every $\epsilon>0$ a positive number $\delta(\epsilon)$ so that $|\lambda'-\lambda''|<\delta(\epsilon)$ implies $|s_k(x,\lambda')-s_k(x,\lambda'')|<\epsilon$ $(k=1,2,\cdots,n)$, provided $\mu\leqq\lambda'<\lambda''\leqq\lambda$. Then, using (5. 43), we get the evaluation

$$\int_a^b|\Xi(y;\delta')-\Xi(y;\delta'')|^2dy\leqq 4n\epsilon^2\sum_{k=1}^n\|\xi^k(\Delta)\|^2,\qquad (\delta',\delta''<\delta(\epsilon)).$$

This shows that, for $\delta\to0$, the function $\Xi(y;\delta)$ converges to a square summable function $\Xi(y)$ in the mean. The mean convergence implies further the existence of a subsequence $\{\Xi(y,\delta_m)\}$, $\delta_m\to0$, converging to $\Xi(y)$ almost everywhere in (a,b). On the other hand, we infer by (5. 49) readily $\lim_{\delta\to0}\Xi(y;\delta)=E(x,y;\Delta)$. Hence $E(x,y;\Delta)$ coincides with $\Xi(y)$ almost everywhere in (a,b); thus we obtain $\int_a^b|E(x,y;\Delta)|^2dy<+\infty$ and

$$\int_a^b|\Xi(y;\delta)-E(x,y;\Delta)|^2dy\to0\qquad(\delta\to0). \tag{5.50}$$

Now using (5. 42) and (5. 50), we conclude from (5. 36)

$$u(x,\Delta)=\lim_{\delta\to0}\sum_m\sum_j s_j(x,\lambda_m)u^j(\Delta_m)=\lim_{\delta\to+0}(u,\sum_m\sum_j s_j(x,\lambda_m)\xi^j(\Delta_m))$$
$$=\lim_{\delta\to+0}(u,\Xi(\delta))=\int_a^bE(x,y;\Delta)u(y)\,dy,$$

proving (5. 20).

Finally the formula (5.21) follows immediately from (5.20), while (5.22) is equivalent to (5.19). Thus the spectral theorem is proved for the special case under consideration.

Now we shall treat the *general case.* For that purpose we denote the special fundamental solutions $s_j(x,l)$ and corresponding $M^{jk}(l), \rho^{jk}(\lambda)$ mentioned above by $s_j^0(x,l)$, $M^{0jk}(l)$, $\rho^{0jk}(\lambda)$, respectively. Then an arbitrary regular system of fundamental solutions $s_j(x,l)$ is obtained from $s_j^0(x,l)$ by a unimodular transformation

$$\text{(U)} \qquad s_j(x,l) = \sum_{k=1}^{n} U_j^k(l) s_k^0(x,l),$$

where $U_j^k(l)$ are entire functions of l satisfying $U_j^k(\bar{l}) = \bar{U}_j^k(l)$. By the transformation (U), the characteristic matrix $M^{jk}(l)$ is transformed as a contravariant tensor, i. e.

$$(5.51) \qquad M^{jk}(l) = \sum_{h,m} V_h{}^j(l)\, V_m{}^k(l) M^{0hm}(l),$$

where $V_j^k(l)$ means the inverse matrix of $U_j^k(l)$; $V_j^k(l)$ are also entire functions of l and satisfy $V_j^k(\bar{l}) = \bar{V}_j^k(l)$.

Let us assume now that $d\rho^{jk}(\lambda)$ constitutes a contravariant tensor, and *define* $\rho^{jk}(\lambda)$ *by the formula*

$$(5.52) \qquad \rho^{jk}(\lambda) = \int_{+0}^{\lambda} \sum_{h,m} V_h{}^j(\lambda)\, V_m{}^k(\lambda)\, d\rho^{0hm}(\lambda).$$

The matrix $\rho(\lambda) = (\rho^{jk}(\lambda))$ thus defined is also real and symmetric, and, as a function of λ, $\rho(\lambda)$ is continuous on the right and monotone non-decreasing. Furthermore *we have*

$$\int_{\Delta} \sum_{j,k} s_j(x,\lambda) s_k(y,\lambda)\, d\rho^{jk}(\lambda) = \int_{\Delta} \sum_{j,k} s_j^0(x,\lambda) s_k^0(y,\lambda)\, d\rho^{0jk}(\lambda),$$

proving the formulae (5.20) and (5.21). To prove the spectral theorem, it is sufficient therefore to verify that the matrix $\rho^{jk}(\lambda)$ thus defined is related to $M^{jk}(l)$ by (5.19).

As to the special matrices $M^{0jk}(l)$, $\rho^{0jk}(\lambda)$, we have the relation (5.48). From this we can deduce the formula

$$(5.53) \qquad M^{0jk}(l) - M^{0jk}(m) = \int_{-\infty}^{+\infty} \{(\lambda - l)^{-1} - (\lambda - m)^{-1}\} d\rho^{0jk}(\lambda)$$
$$(\Im l \neq 0, \Im m \neq 0),$$

where the integral converges absolutely.

For an arbitrary positive number σ, we obtain from (5.53) the expression

(5.54) $$M^{0jk}(l) = \int_{-\sigma}^{\sigma} (\lambda - l)^{-1} d\rho^{0jk}(\lambda) + R_\sigma^{0jk}(l),$$

where the *residual term* $R_\sigma^{0jk}(l)$ is holomorphic with respect to l except for real l such that $l \leqq -\sigma$ or $l \geqq \sigma$. Corresponding to (5.54), we put

(5.55) $$M^{jk}(l) = \int_{-\sigma}^{\sigma} (\lambda - l)^{-1} d\rho^{jk}(\lambda) + R_\sigma^{jk}(l).$$

Then, inserting (5.54) in (5.51) and using (5.52), we obtain

$$R_\sigma^{jk}(l) = \int_{-\sigma}^{\sigma} \sum_{h,m} (\lambda - l)^{-1} \{V_h{}^j(l) V_m{}^k(l) - V_h{}^j(\lambda) V_m{}^k(\lambda)\} d\rho^{0hm}(\lambda)$$
$$+ \sum_{h,m} V_h{}^j(l) V_m{}^k(l) R_\sigma^{0hm}(l),$$

proving that $R_\sigma^{jk}(l)$ is also holomorphic with respect to l except for real l such that $l \leqq -\sigma$ or $l \geqq \sigma$. Furthermore from (5.10) follows

(5.56) $$R_\sigma^{jk}(\bar{l}) = \bar{R}_\sigma^{jk}(l).$$

To deduce (5.19) from (5.55), choose σ so large that $\sigma > |\lambda| + 1$. Then, since $R_\sigma^{jk}(l)$ are regular at real μ such that $|\mu| < |\lambda| + 1$, we infer, by (5.56), $\Im R_\sigma^{jk}(\mu + i\epsilon) \to 0$ $(\epsilon \to +0)$ for $|\mu| < |\lambda| + 1$. Using this, we conclude

$$\lim_{\epsilon \to +0} \int_\delta^{\lambda+\delta} \Im M^{jk}(\lambda + i\epsilon)\, d\lambda = \lim_{\epsilon \to +0} \int_\delta^{\lambda+\delta} d\lambda\, \Im \int_{-\sigma}^{\sigma} (\mu - \lambda - i\epsilon)^{-1} d\rho^{jk}(\mu)$$
$$= \tfrac{1}{2}\pi[\rho^{jk}(\lambda + \delta - 0) + \rho^{jk}(\lambda + \delta) - \rho^{jk}(\delta - 0) - \rho^{jk}(\delta)],$$

yielding immediately (5.19). Thus the spectral theorem is completely proved.

6. Expansion theorem. We introduce for two arbitrary λ-measurable vector functions $\omega(\lambda) = (\omega_1(\lambda), \cdots, \omega_n(\lambda))$, $\chi(\lambda) = (\chi_1(\lambda), \cdots \chi_n(\lambda))$ the notion of their inner product defined by

$$(\omega, \chi) = \int_{-\infty}^{+\infty} \sum_{j,k} \omega_j(\lambda) \bar{\chi}_k(\lambda)\, d\rho^{jk}(\lambda).$$

Then we have $(\omega, \omega) \geqq 0$; $\| \omega \| = (\omega, \omega)^{\frac{1}{2}}$ can be considered therefore as the *norm* of ω. The set Ω of all λ-measurable vector functions ω with $\| \omega \| < +\infty$ constitutes a Hilbert space, if one identifies two functions ω, χ such that $\| \omega - \chi \| = 0$. Now we shall prove

THEOREM 6.1. (EXPANSION THEOREM).[24] *For every function $u(x)$ belonging to $\mathfrak{H}$, the integral*

[24] Cf. Weyl, [11], [12], Titchmarsh, [9], Chap. III. For series expansions associated with ordinary differential equations of any order, see Birkhoff, [1], Tamarkin, [8].

$$(6.1) \qquad \omega_j(\lambda) = \int_a^b s_j(x, \lambda) u(x) dx$$

converges in the sense of the norm in Ω *and, by means of* $\omega_j(\lambda)$ *thus defined,* $u(x)$ *is expanded as*

$$(6.2) \qquad u(x) = \int_{-\infty}^{+\infty} \sum_{j,k} s_j(x, \lambda) \omega_k(\lambda) d\rho^{jk}(\lambda),$$

where the integral converges in the mean; furthermore we have the "Parseval formula"

$$(6.3) \qquad \| \omega \| = \| u \| .$$

Conversely, for every $\omega \varepsilon \Omega$, *the integral* (6.2) *converges in the mean and, by means of* $u(x)$ *defined by* (6.2), ω *is represented as* (6.1).

Proof. Choose t, z such that $a < t < z < b$ arbitrarily and put $u^*(x) = u(x)$ (for $t \leqq x \leqq z$), $= 0$ (for $x < t$ or $x > z$). Then the integrals

$$(6.4) \qquad \omega^*_j(\lambda) = \int_t^z s_j(x, \lambda) u(x) dx = \int_a^b s_j(x, \lambda) u^*(x) dx$$

converge absolutely and represent continuous functions of λ. Hence, applying Fubini's theorem, we get, from (5.20),

$$(6.5) \qquad E(\Delta) u^*(x) = \int_\Delta \sum_{j,k} s_j(x, \lambda) \omega^*_k(\lambda) d\rho^{jk}(\lambda),$$

$$(6.6) \qquad \int_\Delta \omega^*_j(\lambda) \bar{\omega}^*_k(\lambda) d\rho^{jk}(\lambda) = \| E(\Delta) u^* \|^2,$$

for arbitrary finite intervals $\Delta = (\mu, \lambda]$. Making $\mu \to -\infty$, $\lambda \to +\infty$, we get from (6.6) the identity $\| \omega^* \| = \| u^* \|$; thus the linear transformation $u^* \to \omega^*$ defined by (6.4) is isometric. Now we have $\| u^* - u \| \to 0$ $(t \to a, z \to b)$. Hence there exists an element ω in Ω such that $\| \omega^* - \omega \| \to 0$ $(t \to a, z \to b)$, proving the convergence of the integral (6.1) in the sense of the norm in Ω and the Parseval formula (6.3). To prove that $u(x)$ is represented as (6.2) by means of ω thus defined, we consider the integral

$$I_\Delta(x) = \int_\Delta \sum_{j,k} s_j(x, \lambda) \omega_k(\lambda) d\rho^{jk}(\lambda),$$

which evidently converges absolutely. Comparing this with (6.5), we obtain

$$| E(\Delta) u^*(x) - I_\Delta(x) |^2 \leqq \| \omega^* - \omega \|^2 \cdot \int_\Delta s_j(x, \lambda) s_k(x, \lambda) d\rho^{jk}(\lambda) ;$$

this proves $| E(\Delta)u^*(x) - I_\Delta(x) | \to 0$ $(t \to a, z \to b)$, where the convergence is uniform in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$. On the other hand, $E(\Delta)u^*$ converges to $E(\Delta)u$ in the mean. Hence we get

$$E(\Delta)u(x) = \int_\Delta \sum_{j,k} s_j(x, \lambda)\omega_k(\lambda)\, d\rho^{jk}(\lambda), \tag{6.9}$$

yielding immediately (6.2).

To prove the converse proposition, put for an arbitrary element ω of Ω

$$\omega_\Delta(\lambda) = \omega(\lambda) \quad (\text{for } \lambda \in \Delta), \quad = 0 \quad (\text{for } \lambda \notin \Delta),$$

$\Delta = (\mu, \lambda]$ being an arbitrary finite interval. Then the integral

$$u_\Delta(x) = \int_\Delta \sum_{j,k} s_j(x, \lambda)\omega_k(\lambda)\, d\rho^{jk}(\lambda) = \int_{-\infty}^{+\infty} \sum_{j,k} s_j(x, \lambda)\omega_{\Delta k}(\lambda)\, d\rho^{jk}(\lambda) \tag{6.10}$$

converges absolutely and represents a continuous function of x in $a < x < b$. Obviously $\omega_\Delta(\lambda)$ can be approximated by step functions

$$\tilde{\omega}_k(\lambda) = \sum_{m=1}^{h} \gamma_{km} \cdot c(\lambda, \Delta_m)$$

so that $\| \tilde{\omega} - \omega_\Delta \| \to 0$, where Δ_m are mutually disjoint subintervals of Δ and $c(\lambda, \Delta_m)$ denote characteristic functions of Δ_m. As to $\tilde{\omega}$, we have

$$\tilde{u}(x) = \int_{-\infty}^{+\infty} \sum_{j,k} s_j(x, \lambda)\tilde{\omega}_k(\lambda)\, d\rho^{jk}(\lambda) = \sum_m \sum_k \gamma_{km} \cdot \xi^k(x, \Delta_m).$$

Using (5.43) and (5.44), we conclude from this that

$$\| \tilde{u} \|^2 = \sum_m \sum_{j,k} \gamma_{jm}\bar{\gamma}_{km}\rho^{jk}(\Delta_m) = \| \tilde{\omega} \|^2;$$

thus the correspondence $\tilde{\omega} \to \tilde{u}$ is (linear and) isometric. Consequently $\tilde{u}$ converges in the mean to an element u_0 of $\mathfrak{H}$ when $\tilde{\omega}$ approaches to ω_Δ in the sense of the norm. On the other hand, the inequality

$$| \tilde{u}(x) - u_\Delta(x) |^2 \leqq \| \tilde{\omega} - \omega_\Delta \|^2 \cdot \int_\Delta \sum_{j,k} s_j(x, \lambda)s_k(x, \lambda)\, d\rho^{jk}(\lambda)$$

shows that $\tilde{u}(x)$ converges to $u_\Delta(x)$ uniformly in every closed interval $[x_1, x_2]$, $a < x_1 < x_2 < b$, when $\tilde{\omega}$ approaches to ω_Δ. Hence u_Δ coincides with u_0 and therefore $\| u_\Delta \| = \| \omega_\Delta \|$; thus the linear transformation $\omega_\Delta \to u_\Delta$ defined by (6.10) is isometric, proving that the integral (6.2) converges in the mean and that the linear transformation $\omega \to u$ defined by (6.2) is isometric. Now it is obvious that ω is represented as (6.1) by means of $u(x)$ defined by (6.2).

7. Simultaneous differential equations. The above results can be easily extended to the case of simultaneous differential equations. Consider a differential operator

$$L = P_0(x)(d/dx)^h + P_1(x)(d/dx)^{h-1} + \cdots + P_h(x)$$

defined in (a, b) with *matrix* coefficients $P_m(x) = (p_{m\iota\kappa}(x))$, $\iota, \kappa = 1, 2, \cdots, \eta$, operating on vector functions

$$u(x) = (u_1(x), u_2(x), \cdots, u_\eta(x)),$$

where each $p_{m\iota\kappa}(x)$ is a real valued continuous function defined in (a, b) having continuous derivatives up to the order $h - m$ and

$$\det P_0(x) > 0 \ [\text{or} < 0], \qquad (a < x < b). \tag{7.1}$$

The product $n = \eta h$ will be called the "*rank*" of L. Assume furthermore that L coincides with its Lagrange adjoint

$$L^* = (-1)^h(d/dx)^h P_0^*(x) + (-1)^{h-1}(d/dx)^{h-1}P_1^*(x) + \cdots + P_h^*(x),$$

$P_m^*(x)$ being the transposed matrices of $P_m(x)$. In order that L coincides with L^* it is necessary that $P_0(x)$ satisfies $P_0(x) = (-1)^h P_0^*(x)$, which is compatible with (7.1) only if ηh is even. Thus, in the present case, *the order h of L may be odd, whereas the rank $n = \eta h$ must be even*:

$$n = \eta h = 2\nu. \tag{7.2}$$

Now, using the abbreviation

$$u(x)v(x) = \sum_{\kappa=1}^{n} u_\kappa(x) v_\kappa(x),$$

we have *Green's formula*

$$\int_y^x (L[u]v - uL[v])\,dx = [uv](x) - [uv](y), \tag{7.3}$$

where $[uv]$ means the skew-symmetric bilinear form

$$[uv](x) = \sum_{\iota,\kappa} \sum_{j+k \leqq h-1} B_{jk\iota\kappa}(x) u_\iota^{(j)}(x) v_\kappa^{(k)}(x),$$

with the coefficients

$$B_{jk\iota\kappa}(x) = (-1)^{h+j+1} \sum_{m=0}^{h-j-k-1} (-1)^m C_k^{h-j-m-1} p_{m\kappa\iota}^{(h-j-1-k-m)}(x).$$

For $j+k=h-1$, we have

$$B_{jk\iota\kappa}(x)=(-1)^k p_{0\kappa\iota}(x), \qquad (j+k=h-1). \tag{7.4}$$

Hence $[uv](x)$ is a *non-degenerate bilinear form* of two vectors

$$(u_1, u'_1, \cdots, u_1^{(h-1)}, u_2, u'_2, \cdots, u_\eta^{(h-1)}), \qquad (v_1, v'_1, \cdots, v_2, \cdots, v_\eta^{(h-1)}).$$

By means of this bilinear form, we define the "*Wronskian*" of n functions $r, s, t, u, \cdots, v, w$ as

$$[rt\cdots vsu\cdots w]=(1/2^\nu\nu!)\Sigma\pm[rs][tu]\cdots[vw],$$

where $\Sigma\pm$ means the alternating sum extending over all $n!$ permutations of the functions $r, s, t, u, \cdots, v, w$.

Now that it has been established that the bilinear form $[uv](x)$ is non-degenerate, all of our arguments expounded in Sections 2 to 6 can be applied to the present case without essential modifications, since they are entirely based on the bilinear form $[uv](x)$ and Green's formula. It will be sufficient to notice that we have to make the following modifications: First, in Definition 2.1, the functions $s_j(x,l)$ are to be replaced by the vector functions $s_j(x,l)=(s_{j1}(x,l), s_{j2}(x,l), \cdots, s_{j\eta}(x,l))$. Secondly, the inner product (u,v) is to be defined as

$$(u,v)=\int_a^b \sum_\kappa u_\kappa(x)\bar{v}_\kappa(x)\,dx$$

and, correspondingly, $\mathfrak{H}$ will denote the Hilbert space consisting of all measurable *vector functions* u with $(u,u)<+\infty$.

After these modifications, the same arguments as expounded in Sections 2 to 6 lead to the following results: Fix a regular system of fundamental solutions $s_j(x,l)$ $(j=1,2,\cdots,n)$ and associate with every general solution $w(x,l,f)=\Sigma f^j s_j(x,l)$ of the simultaneous differential equations $L[w]=lw$ the point (f) in the $(n-1)$-dimensional projective space $\mathfrak{P}$. Put

$$\mathfrak{m}_b(l)=\{(f);\ \int_c^b \sum_\kappa |w_\kappa(x,l,f)|^2 dx<+\infty\},$$

$$\mathfrak{m}_a(l)=\{(f);\ \int_a^c \sum_\kappa |w_\kappa(x,l,f)|^2 dx<+\infty\}.$$

Then the dimensions of $\mathfrak{m}_a(l)$ and $\mathfrak{m}_b(l)$ do not depend on l and are not less than $\nu-1$, provided $\Im l\neq 0$. Putting

$$\tau_a=\dim\mathfrak{m}_a(l)-\nu+1, \qquad \tau_b=\dim\mathfrak{m}_b(l)-\nu+1,$$

we obtain therefore two non-negative integers τ_a, τ_b which are characteristic for L. Obviously the integers τ_a, τ_b do not exceed ν. Assume now the linearly independent boundary conditions

$$[\phi_{aj}u](a) = 0 \quad (j = 1, 2, \cdots, \tau_a), \qquad [\phi_{bj}u](b) = 0 \quad (j = 1, 2, \cdots, \tau_b),$$

as given, where ϕ_{aj}, ϕ_{bj} are *real* functions belonging to "$\mathfrak{D}$" such that $[\phi_{aj}\phi_{ak}](a) = 0$, $[\phi_{bj}\phi_{bk}](b) = 0$. Then, under these boundary conditions, L becomes a self-adjoint operator, which will be denoted by H. The subspaces

$$\mathfrak{p}_a(l) = \{(f) ; f \varepsilon \mathfrak{m}_a(l), [\phi_{aj}w(l, f)](a) = 0 \ (1 \leqq j \leqq \tau_a)\},$$

$$\mathfrak{p}_b(l) = \{(f) ; f \varepsilon \mathfrak{m}_b(l), [\phi_{bj}w(l, f)](b) = 0 \ (1 \leqq j \leqq \tau_b)\}$$

of $\mathfrak{P}$ will be called the *characteristic spaces* with respect to H. $\mathfrak{p}_a(l)$ and $\mathfrak{p}_b(l)$ have the dimension $\nu - 1$ and have no common point. By the *characteristic matrix* of H we shall mean the matrix $M^{jk}(l)$ defined by the relation

$$\sum_m \sum_j M^{jk}(l)[s_j(l)s_m(l)] \cdot f^m = f^k \text{ (for } f \varepsilon \mathfrak{p}_a(l)), = 0 \text{ (for } f \varepsilon \mathfrak{p}_b(l)).$$

As a function of l, the matrix $M^{jk}(l)$ is holomorphic in each half plane $\Im l > 0$, $\Im l < 0$, and $M^{jk}(\bar{l}) = \bar{M}^{jk}(l)$. Now we have

THEOREM 7.1. (SPECTRAL THEOREM). *For every real number λ, there exists the limit*

$$\rho^{jk}(\lambda) = \lim_{\delta \to +0} \lim_{\epsilon \to +0} (1/\pi) \int_\delta^{\lambda + \delta} \Im M^{jk}(\lambda + i\epsilon) d\lambda.$$

The matrix $\rho(\lambda) = (\rho^{jk}(\lambda))$ thus defined is real and symmetric. As a function of λ, $\rho(\lambda)$ is continuous on the right and monotone non-decreasing in the sense that, for $\mu < \lambda$, the symmetric matrix $\rho(\lambda) - \rho(\mu)$ is positive semi-definite. Let

$$H = \int_{-\infty}^{+\infty} \lambda dE(\lambda) \quad \cdot$$

be the spectral decomposition of H and put, for every finite interval $\Delta = (\mu, \lambda]$, $E(\Delta) = E(\lambda) - E(\mu)$. Then $E(\Delta)$ is represented as the integral operator with the matrix kernel

$$E_{\iota\kappa}(x, y, \Delta) = \int_\Delta \sum_{j,k} s_{j\iota}(x, \lambda) s_{k\kappa}(y, \lambda) d\rho^{jk}(\lambda)$$

in the sense that

$$E(\Delta)u_\iota(x) = \int_a^b \sum_\kappa E_{\iota\kappa}(x, y, \Delta) u_\kappa(y) dy$$

for every $u \in \mathfrak{H}$, where

$$\int_a^b \sum_\kappa | E_{\iota\kappa}(x, y, \Delta) |^2 dy < +\infty.$$

Finally, as in **6**, we define the *norm* $\| \omega \|$ of an arbitrary λ-measurable vector function $\omega(\lambda) = (\omega_1(\lambda), \cdots, \omega_n(\lambda))$ by

$$\| \omega \|^2 = \int_{-\infty}^{+\infty} \sum_{j,k} \omega_j(\lambda) \overline{\omega}_k(\lambda) d\rho^{jk}(\lambda).$$

Then we have

THEOREM 7.2. (EXPANSION THEOREM). *For every vector function $u(x)$ belonging to $\mathfrak{H}$, the integral*

$$\omega_k(\lambda) = \int_a^b \sum_\kappa s_{k\kappa}(x, \lambda) u_\kappa(x) dx$$

converges in the sense of the norm defined above and, by means of $\omega_k(\lambda)$ thus defined, the vector function $u(x)$ can be expanded as

$$u_\kappa(x) = \int_{-\infty}^{+\infty} \sum_{j,k} s_{j\kappa}(x, \lambda) \omega_k(\lambda) d\rho^{jk}(\lambda),$$

where the integral converges in the sense of the norm in $\mathfrak{H}$; furthermore we have the "Parseval formula": $\| u \| = \| \omega \|$.

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ON THE ESSENTIAL SPECTRA OF SINGULAR EIGENVALUE PROBLEMS.*

By Philip Hartman and Aurel Wintner.

1. Let λ be a real parameter, and let $p(t) > 0$ and $q(t)$, where $0 \leqq t < \infty$, be a pair of real-valued, continuous functions for which the differential equation

$$(1) \qquad (px')' + (q + \lambda)x = 0,$$

where $' = d/dt$, becomes of *Grenzpunkt* type in the sense of Weyl [4]. This proviso, which will not be repeated below, means that (1) possesses some solution $x(t) = x_\lambda(t)$ violating the L^2-condition

$$(2) \qquad \int_0^\infty x^2(t)dt < \infty$$

(for some λ, but then for every λ; cf. [4], p. 238). Correspondingly, (1) and any boundary condition of the form

$$(3_\alpha) \qquad x(0)\cos\alpha + p(0)x'(0)\sin\alpha = 0, \qquad (p(0) > 0),$$

where the angle α is arbitrary, determine an eigenvalue problem.

Let S_α denote the λ-set representing the spectrum of the eigenvalue problem defined by (1) and (3_α). It is known that every S_α is a closed set. According to Weyl ([4], p. 251), the set of the cluster points of S_α is independent of α; it can therefore be denoted simply by S'. The invariant λ-set S' will be referred to as the essential spectrum of (1).

It was proved in [2] that (1) has a non-trivial solution satisfying (2) whenever λ is not in S'. On the other hand, since S' is a closed set, a value $\lambda = \lambda^0$ cannot be in the complement of S' unless the same is true of every λ which is close enough to λ^0. One might expect that these two facts together suffice for the characterization of the complement of S' (and, therefore, of S' itself), but we could not prove this (cf. [3], Appendix). Also that characterization of S' which results, in terms of the zeros of the solutions of

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(1), from Theorem (I) in [1] is a characterization depending not on a *single* value, λ^0, of λ but is such as to involve an entire neighborhood of λ^0.

In what follows, a theorem will be proved which allows the characterization of a point of S' in such a way as to involve *only that point*. The resulting characterization of points of S' will, in addition, be such as to involve only the differential equation and *no specification of a boundary condition*, (3_α). In fact, the following theorem will be proved:

(*) *A real* $\lambda = \lambda_0$ *is in the essential spectrum of* (1) *if and only if there exists some* $g = g(t)$ *which is real-valued and continuous for* $0 \leqq t < \infty$, *satisfies the* L^2*-condition*

$$\int_0^\infty g^2(t)\,dt < \infty, \tag{4}$$

and is such that the corresponding inhomogeneous differential equation

$$(px')' + (q + \lambda_0)x = g \tag{5}$$

does not possess any solution $x = x(t)$ *satisfying the* L^2*-condition* (2).

2. The proof of this theorem will depend on a Lebesgue-Toeplitz "norm construction," similar to that used in [5] for the characterization of the points of S_α in terms of (5), with a g satisfying (4). The construction applied in [5] will have to be modified so as to replace the spectrum S_α, where α is fixed in (3_α), by the essential spectrum S', which is identical with the λ-set consisting of the common part of every S_α, where $0 \leqq \alpha < \pi$.

The latter characterization of S' is clear from the fact that if $\lambda = \lambda_0$ belongs to two of the spectral sets, say S_α and S_β, where $\alpha \neq \beta \pmod{\pi}$, then $\lambda = \lambda_0$ certainly is a cluster point of at least one of these sets (and, therefore, of both). For otherwise $\lambda = \lambda_0$ is in the point spectrum of two distinct boundary conditions, (3_α) and (3_β). But this is impossible, since (1) does not have (for any λ) two linearly independent solutions, $x = x_\alpha(t)$ and $x = x_\beta(t)$, satisfying (2).

According to Weyl [4], p. 251, the situation belonging to a *fixed* boundary condition is as follows:

(i) A $\lambda = \lambda_0$ is not in S_α if and only if there belongs to every continuous g satisfying (4) *one and only one* solution $x = x_g$ of (5) satisfying (3_α) and (2).

In contrast, (*) can be restated as follows:

(ii) A $\lambda = \lambda_0$ is not in S' if and only if there belongs to every continuous g satisfying (4) *at least one* solution $x = x_g$ of (5) satisfying (2).

Since the complement of the λ-set S' is identical with the sum of the complements of all sets S_α, where $0 \leqq \alpha < \pi$, the assertion of (*) is equivalent to the statement that the elimination of the parameter α from (i) must lead to (ii). In this regard, it is easy to see that the italicized specification in (ii) cannot be replaced by that in (i).

3. The sufficiency of the condition claimed by (*) for a point λ_0 of S' follows at once. For, if there exists a continuous g, satisfying (4), for which (5) fails to have a solution, x, satisfying (2), then it follows from (i) that λ_0 is in S_α, no matter what α may be. In other words, λ_0 is in the common part of all spectra S_α, that is, in S', as claimed by (*).

The converse assertion of (*) is substantially equivalent to the following lemma:

(§) *If* (1) *has no solution* $x(t) \not\equiv 0$ *of class* $L^2(0,\infty)$ *when* $\lambda = \lambda_0$, *then there exists a continuous* $g(t)$, *of class* $L^2(0,\infty)$, *for which* (5) *has no solution* $x(t)$ *of class* $L^2(0,\infty)$.

In fact, if this lemma, (§), is granted, then the necessity of the condition claimed by (*) for a point of S' can be concluded as follows:

Suppose that (5) has, for every continuous $g(t)$ satisfying (4), at least one solution satisfying (2). It then follows from (§) that (1) has, when $\lambda = \lambda_0$, a non-trivial solution of class $L^2(0,\infty)$. Let γ be the angle α (determined mod π) for which this solution satisfies (3_α), and let $x = y(t)$ denote this solution.

For a given g, let $x = x_g(t)$ be a solution of (5) which is of class $L^2(0,\infty)$ (by assumption, there exists such an x_g for every g). Then it is clear from the definition of $y(t)$ that, if c is any constant, the sum $x(t,c) = x_g(t) + cy(t)$ is again an x_g, and all x_g's are of this form. On the other hand, it follows from the definition of the angle γ that, if α is any angle distinct from γ (mod π), then there exists a unique $c = c(\alpha)$ corresponding to which the sum $x(t,c)$ will satisfy (3_α). In fact, since $x(t,c) = x_g(t) + cy(t)$, this $c = c(\alpha)$ is given by the ratio

$$c = -\{x_g(0)\cos\alpha + p(0)x'_g(0)\sin\alpha\}/\{y(0)\cos\alpha + p(0)y'(0)\sin\alpha)\}.$$

The denominator is distinct from 0, since, on the one hand, $\alpha \not\equiv \gamma$ (mod π) and, on the other hand, $y(t)$ belongs to the boundary conditions (3_γ).

Accordingly, if $g(t)$ is any continuous function satisfying (4), and if α is any angle distinct from $\gamma \pmod{\pi}$, then (5) has a unique solution satisfying (2) and (3_α). It follows therefore from (i) that λ_0 is not in S_α for any α distinct from $\gamma \pmod{\pi}$. Since S' is the common part of all spectra S_α, this implies that λ_0 cannot be in S', as claimed by (*).

This proves that (*) is true if (§) is granted.

4. The proof of (§) proceeds as follows:

For the given value, λ_0, of λ, choose two (real-valued) linearly independent solutions, say $x = x_1(t)$ and $x = x_2(t)$, of (1). Since their Wronskian, $x_2x_1' - x_1x_2'$, when multiplied by the first coefficient function, $p = p(t)$, of (1), becomes a non-vanishing constant, it can be assumed that

$$p(t)\{x_2(t)x_1'(t) - x_1(t)x_2'(t)\} = 1. \tag{7}$$

On the range $0 \leqq a < b < \infty$, define a function $\mu = \mu(a, b)$ by placing

$$\mu(a, b) = \min_{0 \leqq \theta < \pi} \int_a^b \{x_1(t)\cos\theta + x_2(t)\sin\theta\}^2 d\theta. \tag{8}$$

Then $\mu(a, b)$ is a non-decreasing function of b. Hence, the limit $\mu_a = \mu(a, \infty)$ exists for every non-negative a, with the understanding that the possibility $\mu_a = \infty$ is not excluded. It also follows that $\mu(a, b) \leqq \mu_a$. Actually, $\mu(a, b) < \mu_a$ (whether $\mu_a = \infty$ or $\mu_a < \infty$). In fact, since $x_1(t)$ and $x_2(t)$ are linearly independent, it is clear from (8) that $\mu(a, b)$ is a strictly increasing function of b.

It will be shown that, for every fixed a,

$$\mu(a, b) \to \infty \text{ as } b \to \infty, \tag{9}$$

if λ_0 satisfies the assumption of (§).

Suppose that (9) is false. Then $\mu_a < \infty$ (for some a). But it will be shown that there exists a $\theta^* = \text{const.}$ satisfying

$$\int_a^T \{x_1(t)\cos\theta^* + x_2(t)\sin\theta^*\}^2 dt \leqq \mu_a \tag{10}$$

for every $T > a$, and (10) implies that $x(t) = x_1(t)\cos\theta^* + x_2(t)\sin\theta^*$ is of class $L^2(0, \infty)$. In addition, this $x(t)$ is a solution of (1) for $\lambda = \lambda_0$, and it does not vanish identically, since $x_1(t)$ and $x_2(t)$ are linearly independent. Consequently, the assumption of (§) is contradicted.

Accordingly, (9) will be proved if it is shown that there exists a θ^* satisfying (10). But such a θ^* can be constructed as follows:

According to (8), there exists on the interval $0 \leqq \theta < \pi$ at least one $\theta = \theta(a, b)$ for which the integral occurring in (8) attains the value $\mu(a, b)$. Choose $b = n$, where n is any positive integer exceeding a fixed value of a, and put $\theta_n = \theta(a, n)$. Then, if $a < T < n$, it is clear that

$$\int_a^T \{x_1(t)\cos\theta_n + x_2(t)\sin\theta_n\}^2 dt < \mu(a, n) < \mu_a.$$

Hence, in order to conclude the existence of a θ^* satisfying (10), it is sufficient to observe that, since $0 \leqq \theta_n < \pi$, the values θ_n must have at least one cluster value. In fact, the latter can then be chosen to be θ^*.

This concludes the proof of (9).

5. In view of (9) and (8), there exists a sequence of t-values $0 = T_0 < T_1 < \cdots$ satisfying $\mu(T_{n-1}, T_n) > 1$ for $n = 0, 1, \cdots$ and $T_n \to \infty$ as $n \to \infty$. In particular

$$(11) \qquad \int_{T_{n-1}}^{T_n} x_2^2(t)\,dt \geqq \mu(T_{n-1}, T_n) > 1, \text{ where } n = 1, 2, \cdots; T_0 = 0.$$

In fact, the first of the inequalities (11) follows by observing that, in view of the definition (8), the integral of $x_2^2(t)$ over an interval $a \leqq t \leqq b$ cannot be less than $\mu(a, b)$.

In terms of the sequence $T_0, T_1, \cdots$, define for $0 \leqq t < \infty$ a function $g(t)$, which is continuous except for possible jumps at $t = T_n$, by placing

$$(12) \qquad g(t) = C_n x_2(t) \text{ if } T_n \leqq t < T_{n+1},$$

where $n = 0, 1, 2, \cdots$, and, according as $n = 2m$ or $n = 2m + 1$,

$$(13) \qquad 1/C_{2m} = (m+1)\{\int_{T_{2m}}^{T_{2m+1}} x_2^2(t)\,dt\}^{\frac{1}{2}}, \qquad C_{2m+1} = 0.$$

For this $g(t)$, the value of the integral (2) is seen to be $\Sigma(m+1)^{-2}$. Hence, this $g(t)$ satisfies (2). It will be shown that, for this $g(t)$, the differential equation (5) fails to have any solution satisfying (2). This will complete the proof of (§), provided that $g(t)$ is allowed to have a sequence of jumps, instead of being continuous as specified in (§). But this complication is immaterial, since it will be clear from the proof below that the above $g(t)$ could readily be smoothed out so as to become continuous.

6. Since $x_1(t)$ and $x_2(t)$ are solutions of the homogeneous case, $g(t) \equiv 0$, of (5), it is readily verified from (7) that, if $g(t)$ is any continuous function,

$$(14) \qquad x(t) = x_1(t) \int_0^t x_2(s) g(s) ds - x_2(t) \int_0^t x_1(s) g(s) ds$$

is a solution of (5). The same holds if $g(t)$ is not continuous, provided that (5) is replaced (at the discontinuity points, say $t = T_0, T_1, \cdots$, of g) by its integrated form,

$$(5') \qquad p(t)x'(t) - p(0)x'(0) + \int_0^t \{q(s) + \lambda_0\} x(s) ds = \int_0^t g(s) ds.$$

With this understanding, the general solution of (5) is

$$(15) \qquad x(t) = x_1(t)\{c_1 + \int_0^t x_2(s) g(s) ds\} - x_2(t)\{c_2 + \int_0^t x_1(s) g(s) ds\},$$

where c_1, c_2 denote arbitrary constants. In fact, the difference of the functions (14), (15) is the general solution of the homogeneous case, $g(t) \equiv 0$, of (5).

Consequently, what remains to be verified can be formulated as follows: If $g(t)$ is defined by (12) and (13), then the function (15) violates the L^2-condition (2) for *every* choice of the constants c_1, c_2.

First, if t is on an interval $T_n \leqq t < T_{n+1}$ belonging to an odd n, then $g(t) = 0$, by (13) and (12). It follows therefore from (15) that

$$(16) \qquad x(t) = A_m x_1(t) - B_m x_2(t) \text{ if } T_{2m+1} \leqq t < T_{2m+2},$$

where A_m, B_m denote the constants

$$(17) \qquad A_m = c_1 + \int_0^{T_{2m+1}} x_2(s) g(s) ds, \qquad B_m = c_2 + \int_0^{T_{2m+1}} x_1(s) g(s) ds.$$

It is clear from (8) and (16) that

$$(18) \qquad \int_{T_{2m+1}}^{T_{2m+2}} x^2(t) dt \geqq (A_m{}^2 + B_m{}^2) \mu(T_{2m+1}, T_{2m+2}).$$

On the other hand, the second of the inequalities (11) shows that the product to the right of (18) exceeds $(A_m{}^2 + B_m{}^2) \cdot 1$. It follows therefore from (18) that the value of the integral (2) is minorized by the sum of the series

$\Sigma(A_m^2 + B_m^2)$. Consequently, the function (15) cannot satisfy the (L^2)-condition (2) if $\Sigma A_m^2 = \infty$. Accordingly, the proof will be complete if it is ascertained that $\Sigma A_m^2 = \infty$ holds for every choice of c_1, c_2 in (15). But this, and even $A_m \to \infty$ as $m \to \infty$, can be seen as follows:

According to (12) and (17),

$$A_m = c_1 + \sum_{n=1}^{2m+1} C_n \int_{T_{n-1}}^{T_n} x_2^2(t)\,dt.$$

It follows therefore from the second of the relations (13) that $A_m \to \infty$, as $m \to \infty$, must hold if the series

$$\sum_{m=0}^{\infty} C_{2m} \int_{T_{2m}}^{T_{2m+1}} x_2^2(t)\,dt$$

is divergent. But (13) shows that this series must diverge if the series

$$\sum_{m=0}^{\infty} (m+1)^{-1}\{\int_{T_{2m}}^{T_{2m+1}} x_2^2(t)\,dt\}^{\frac{1}{2}}$$

does. Since (11) implies that the latter series is minorized by the series $\Sigma(m+1)^{-1}\cdot 1 = \infty$, the proof is complete.

7. The content of (*) is that the space of functions g which are real-valued, continuous, satisfy (4) and have the property that (5) possesses no solution satisfying (2), is not empty if and only if $\lambda = \lambda_0$ is in the essential spectrum of (1).

This suggests the consideration of a "dual" space of functions g, namely, the space $\Omega = \Omega(\lambda_0)$ of functions g which are real-valued, continuous, satisfy (4) and have the property that, for *every* α, the differential equation (5) possesses a solution $x = x_{a,g}(t)$ satisfying (2) and (3_a). It turns out that, *for every value of* $\lambda = \lambda_0$, *the linear space* $\Omega(\lambda_0)$ *is not empty and is, in fact,* ∞*-dimensional.*

In order to see this, for a given value of $\lambda = \lambda_0$, let $x_1(t)$ and $x_2(t)$ be two solutions of (1) satisfying (7). If a continuous $g(t) = g_T(t)$, where $T > 0$, satisfies $g(t) = 0$ for $t \geqq T$ and

$$(19) \qquad \int_0^T x_1(t)g(t)\,dt = \int_0^T x_2(t)g(t)\,dt = 0,$$

then $g(t)$ belongs to $\Omega(\lambda_0)$. Clearly the set of such functions g is ∞-

dimensional. For a given g, the function $x(t)$ given by (14) is a solution of (5) and satisfies $x(0) = x'(0) = 0$; hence it satisfies (3_a) for every α. On the other hand, if $g(t) = 0$ for $t > T$ and if (19) holds, then $x(t) = 0$ for $t > T$, hence $x(t)$ satisfies (2).

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ON THE EMBEDDING PROBLEM IN DIFFERENTIAL GEOMETRY.*

By Philip Hartman and Aurel Wintner.

1. The theorem. Let $g_{ik} = g_{ik}(u, v)$ be the elements of a 2 by 2, symmetric, positive definite (function) matrix, defined in a neighborhood of $(u, v) = (0, 0)$. The problem to be dealt with concerns the existence of a 2-dimensional surface, in a 3-dimensional Euclidean space, for which $ds^2 = g_{11}du^2 + 2g_{12}du\,dv + g_{22}dv^2$; in other words, with the existence of functions $x = x^j(u, v)$, where $j = 1, 2, 3$, which satisfy the three conditions

$$(1) \qquad \sum_{j=1}^{3} (\partial x^j/\partial u^i)(\partial x^j/\partial u^k) = g_{ik}, \qquad (u^1 = u, u^2 = v),$$

in some vicinity of $(0, 0)$. Earlier writers (cf., e. g., [2], pp. 38-39) have assumed that the g_{ik} are analytic functions of (u, v); so that the Cauchy-Kowalewski theorem on partial differential equations becomes applicable. This assumption will not be made below.

If the g_{ik} are of class C^2, then the Theorema Egregium makes it possible to define the curvature $\kappa = \kappa(u, v) = \kappa(u, v; g_{ik})$ belonging to the given g_{ik}. Using the Frobenius representation of the Gaussian curvature, put

$$(2) \qquad \kappa = -\tfrac{1}{4}\Delta^{-4} D - \tfrac{1}{2}\Delta^{-1}\{[\Delta^{-1}(g_{11v} - g_{12u})]_v - [\Delta^{-1}(g_{12v} - g_{22u})]_u\},$$

where $\Delta = (g_{11}g_{22} - g_{12}{}^2)^{\frac{1}{2}} > 0$; the subscripts u, v denote partial differentiation; D is the 3-rowed determinant in which the first column consists of the numbers g_{11}, g_{12}, g_{22}, and the second and third columns are formed by the partial derivatives of g_{11}, g_{12}, g_{22} with respect to u and v, respectively.

A point (u, v) will be called hyperbolic, parabolic or elliptic according as $\kappa(u, v)$ is negative, zero or positive. A (u, v)-domain will be said to consist of "points of the same type" if all of its points belong to one and the same of these three cases.

If $n \geqq 0$ and $0 < \lambda < 1$, a function is called of class $C^n(\lambda)$ in a domain if it possesses n-th order partial derivatives and these partial derivatives satisfy a uniform Hölder condition of order λ with respect to all of the variables.

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THEOREM. *If $n \geqq 2$ and $0 < \mu < \lambda < 1$, and if the $g_{ik}(u, v)$ are given functions of class $C^n(\lambda)$ and have the property that every point (u, v) in a neighborhood of $(0, 0)$ is of the same type, then there exist three functions $x^j = x^j(u, v)$ of class $C^n(\mu)$ satisfying (1) in a neighborhood of $(0, 0)$.*

The question of existence of functions $x^j = x^j(u, v)$ satisfying (1) is left open in case $(0, 0)$ is a parabolic point which is a cluster point of points of elliptic and/or hyperbolic type.

2. Counter-examples. In a certain sense, the above theorem is the best possible of its type. In fact, if the given functions g_{ik} are of class $C^n(\lambda)$, there need not exist functions $x^j = x^j(u, v)$ of class $C^n(\lambda + \epsilon)$, for any $\epsilon > 0$ satisfying (1). For example, let $n \geqq 2$ and $g_{11} = 1 + v^2 + v^{n+\lambda}$, $g_{12} \equiv 0$, $g_{22} \equiv 1$; so that these functions are of class $C^n(\lambda)$. According to (2), the curvature is

$$\kappa = \tfrac{1}{2} g_{11}^{-1}[2 + c_1 v^{2n+2\lambda-2} + c_2 v^{n+\lambda} + (n + \lambda)(n + \lambda - 1) v^{n+\lambda-2}],$$

where c_1, c_2 are constants. Hence κ is of class $C^{n-2}(\lambda)$, but is not $C^{n-2}(\lambda + \epsilon)$ for any $\epsilon > 0$. On the other hand, if there exist functions $x^j(u, v)$ of class $C^n(\lambda + \epsilon)$, for some $\epsilon > 0$, satisfying (2), it follows from the definition of curvature in terms of the second fundamental form that κ is of class $C^{n-2}(\lambda + \epsilon)$. (Even if $n = 2$, the curvature given by (2) is identical with the Gaussian curvature, defined in terms of the second fundamental form; cf. Section **7** below.)

At first glance, this counter-example seems surprising, since one might expect that, to given g_{ik} of class C^n, there correspond surfaces of class C^{n+1}. The Theorema Egregium indicates, however, the presence of a "paradox," since the curvature, which can be defined in terms of the first and second partial derivatives of the $x^j(u, v)$, is expressed in terms of quantities involving the third order partial derivatives of the $x^j(u, v)$. Actually, a more general formulation of the Theorema Egregium avoids this difficulty. Such a formulation was given by Weyl, [12], pp. 43-44 (cf. van Kampen, [3], p. 135) and will be used in the proof of the Theorem; see Section **7**.

3. The parabolic case. If the $g_{ik}(u, v)$ are of class $C^n(\lambda)$ in a neighborhood of $(0, 0)$, then there exists a pair of functions $u = u(u', v')$, $v = v(u', v')$, of class $C^{n+1}(\lambda)$ in a neighborhood of $(u', v') = (0, 0)$, which have the properties that $u(0, 0) = 0$, $v(0, 0) = 0$, that $\partial(u, v)/\partial(u', v') \neq 0$, and that the line-element belonging to

$$(3)\qquad g'_{km}(u',v')=\sum_{j=1}^{2}\sum_{k=1}^{2}g_{ij}(\partial u^i/\partial u'^k)(\partial u^j/\partial u'^m),\qquad (u'^1,u'^2)=(u',v'),$$

is conformal, that is, $g'_{11}=g'_{22}$ and $g'_{12}\equiv 0$. For the case $n=0$, see Lichtenstein [7]; if $n\geqq 0$, see Lichtenstein [6], also Korn [5]. Clearly, the functions g'_{ik} are of class $C^n(\lambda)$.

If $n\geqq 2$, the invariance of the Gaussian curvature and the Theorema Egregium (that is, (2)) show that

$$\partial^2\Gamma/\partial u'^2+\partial^2\Gamma/\partial v'^2=0,\qquad (e^{\Gamma}=g'_{11}=g'_{22}),$$

since it is assumed that the curvature is identically 0. Since Γ is of class $C^2(\lambda)$ (hence, of class C^2), it follows that Γ is analytic. Accordingly, there exists, in a neighborhood of $(u^*,v^*)=(0,0)$, a pair of analytic functions $u=u'(u^*,v^*)$, $v=v'(u^*,v^*)$ having the properties that $u'(0,0)=0$, $v'(0,0)=0$, that $\partial(u',v')/\partial(u^*,v^*)\neq 0$, and that the line-element belonging to

$$g^*_{km}(u^*,v^*)=\sum_{j=1}^{2}\sum_{i=1}^{2}g'_{ij}(\partial u'^i/\partial u^{*k})(\partial u'^j/\partial u^{*m}),\qquad (u^{*1},u^{*2})=(u^*,v^*)$$

is Euclidean, that is, $g^*_{11}=g^*_{22}\equiv 1$ and $g^*_{12}\equiv 0$.

The composite transformation $(u,v)\rightarrow(u^*,v^*)$ is of class $C^{n+1}(\lambda)$, and has a non-vanishing Jacobian, near $(0,0)$. Hence the inverse transformation, $u^*=u^*(u,v)$, $v^*=v^*(u,v)$, has the same properties. Put $x^1=u^*(u,v)$, $x^2=v^*(u,v)$ and $x^3\equiv 0$. These functions are of class $C^{n+1}(\lambda)$ and satisfy (1), since $du^{*2}+dv^{*2}=g_{11}du^2+2g_{12}du\,dv+g_{22}dv^2$.

This completes the proof of the Theorem in the parabolic case. In fact, the solutions $x^j(u,v)$ of (1) exist, and can be chosen to be of class $C^{n+1}(\lambda)$, in this case (instead of merely of class $C^n(\mu)$, as stated for the general case).

4. Two Lemmas. Before proceeding to the elliptic and hyperbolic cases, two lemmas concerning the existence of solutions for partial differential equations will be proved. Lemma 2 below is suggested by Picard's theorem [10] on the existence of solutions of linear boundary value problems on small domains and by Lichtenstein's theorem ([8], pp. 89-96) on non-linear elliptic partial differential equations involving a small parameter; principles which go back to H. A. Schwarz. Lemma 2 is also related to the procedure used by Weyl ([12], pp. 64-68) for the "embedding in the large" of a first fundamental form near to that of a sphere.

Let $r>0$, $0<\mu<1$, and let $f(u,v)$ be of class $C^0(\mu)$ (that is, let f satisfy a uniform Hölder condition of order μ) in the circle $\mathfrak{C}_r$: $u^2+v^2<r^2$.

By the symbol $|f|_\mu$ will be meant the least (non-negative) number M satisfying both

$$|f(u,v)| \leqq M \text{ and } |f(u+h, v+k) - f(u,v)|(h^2+k^2)^{-\frac{1}{2}\mu} \leqq M$$

for all pairs of distinct points (u,v), $(u+h, v+k)$ in $\mathcal{B}_r$. If a function $w(u,v)$ possesses continuous second order partial derivatives in $\mathcal{B}_r$, put

$$(4) \qquad w_1 = w_{uu}, \quad w_2 = w_{uv}, \quad w_3 = w_{vv}, \quad w_4 = w_u, \quad w_5 = w_v, \quad w_6 = w.$$

If $w(u,v)$ is of class $C^2(\mu)$ in $\mathcal{B}_r$, let $\| w \|$ denote the greatest of the six numbers $|w_1|_\mu, |w_2|_\mu, \cdots, |w_6|_\mu$. For a given f or w of class $C^0(\mu)$ or $C^2(\mu)$ on $\mathcal{B}_R$, the numbers $|f|_\mu$, $\|w\|$ are functions of r (on an interval $0 < r \leqq R$).

LEMMA 1. *In a neighborhood of* $(u,v) = (0,0)$, *let* $a^i = a^i(u,v)$, *where* $i = 1, 2, \cdots, 6$, *be functions of class* C^3 *satisfying*

$$(5) \qquad a^2a^2 - 4a^1a^3 \neq 0,$$

and let $0 < \mu < 1$. *Then there exists a pair of constants* R, M *(depending on* μ*) with the property that if* $0 < r \leqq R$, *and if* $f = f(u,v)$ *is of class* $C^0(\mu)$ *in* $\mathcal{B}_r$, *then there exists a function* $w = w_f = w(u,v)$ *of class* $C^2(\mu)$ *satisfying*

$$(6) \qquad \sum_{i=1}^{6} a^i w_i = f$$

in $\mathcal{B}_r$; *also*

$$(7) \qquad \| w \| \leqq M |f|_\mu;$$

finally, $w = w_f$ *depends linearly on the function* f.

In the applications below, the functions $a^i(u,v)$ will be analytic. In this case, the proof of Lemma 1 shows that if f is of class $C^n(\mu)$, then w_f is of class $C^{n+2}(\mu)$, and that there exists an estimate of the type (7) for higher order partial derivatives.

Proof of Lemma 1 *when* $a^2a^2 - 4a^1a^3 < 0$. In this case, (6) is an inhomogeneous linear partial differential equation of elliptic type. If R is chosen sufficiently small, the function $w = w_f(u,v)$ can be chosen to be the unique solution of class $C^2(\mu)$ of (6) satisfying $w(u,v) = 0$ for $u^2 + v^2 = r^2$; cf. Lichtenstein [8], pp. 91-92 and the references given there. The estimate (7) goes back to Korn [4]. That M can be chosen independent of r (on the range $0 < r \leqq R$) follows from Korn's proof of the existence of M for a fixed r; cf. [7], p. 201, footnote 2.

Proof of Lemma 1 *when* $a^2a^2 - 4a^1a^3 > 0$. In this case, (6) is an inhomogeneous linear differential equation of hyperbolic type. If R is chosen sufficiently small, the function $w = w_f(u, v)$ can be chosen to be the unique solution of class $C^2(\mu)$ of (6) satisfying $w(u, v) = 0$ on suitably fixed arcs through $(0, 0)$. In order to see this, let a transformation $u = u(u', v')$, $v = v(u', v')$, of class C^3 and of non-vanishing Jacobian, be so chosen that, if u, v in (6) are replaced by u', v', then (6), in terms of u', v', appears in the standard normal form

$$w_2 + a^4w_4 + a^5w_5 + a^6w_6 = f_0.$$

If $w = w(u', v')$ denotes that solution of this normal form which satisfies the conditions $w(0, v') \equiv 0$, $w(u', 0) \equiv 0$, then Riemann's integral representation shows that w has the properties claimed in Lemma 1.

Lemma 2. *In the Monge-Ampère differential equation*

$$a(z_1z_3 - z_2{}^2) + \sum_{m=1}^{5}\sum_{k=4}^{5} b_{km}z_kz_m + d = 0, \tag{8}$$

let a, b_{km}, d *be functions of* (u, v) *satisfying, in a neighborhood* $(0, 0)$, *a uniform Hölder condition of order* λ, *and let, in that neighborhood,*

$$ad \neq 0. \tag{9}$$

Then, if $0 < \mu < \lambda < 1$, *there exist functions* $z = z(u, v)$ *of class* $C^2(\mu)$ *satisfying* (8) *in some neighborhood of* $(0, 0)$.

The proof will show that if the functions a, b_{km}, d are of class $C^n(\lambda)$ in a neighborhood of $(0, 0)$, then the solution $z = z(u, v)$ can be chosen of class $C^{n+2}(\mu)$.

Remark. That the existence of solutions of (8) in the small is not trivial can be seen from the fact that there exist comparatively innocent-looking partial differential equations possessing *no solution in the small.* For example, if $F(u)$ is a continuous, nowhere differentiable function and $f(u, v) = F(u + v)$, then the considerations of Perron [9], pp. 550-551, show that $z_u - z_v = f$, a linear partial differential equation of first order, has *no* solution.

In order to systematize the notations, the equation (8) will sometimes be written as

$$\sum_{k=1}^{5}\sum_{k=1}^{5} b_{km}z_kz_m + d = 0, \tag{8 bis}$$

where it is supposed that $b_{km} = b_{mk}$ and $b_{11} = b_{12} = b_{23} = b_{33} = 0$, while $2b_{13} = -b_{22} = a$.

Proof of Lemma 2. Denote the expression on the left of (8) by $\phi(u, v, z_1, z_2, \cdots, z_5)$, and consider the partial differential equation

$$(10) \qquad \phi(0, 0, z_1, z_2, \cdots, z_5) = 0.$$

Let the numbers $z^0{}_1, z^0{}_3$ be chosen in such a way that $\phi(0, 0, z^0{}_1, 0, z^0{}_3, 0, 0) = 0$; such a choice is clearly possible, since $a(0,0) \neq 0$ in (8). Since $d(0,0) \neq 0$, it follows that $z^0{}_1 \neq 0$, $z^3{}_0 \neq 0$. Thus $\partial\phi(0, 0, z^0{}_1, 0, z^3{}_0, 0)/\partial z_1 = a(0,0) z^0{}_3 \neq 0$. Hence, (10) can be written in the form

$$(11) \qquad z_1 - \psi(z_2, z_3, z_4, z_5) = 0$$

in a neighborhood of $(z_1, \cdots, z_5) = (z^0{}_1, 0, z^0{}_3, 0, 0)$. The function ψ is analytic in its four variables. It follows therefore from the Cauchy-Kowalewski theorem that (11) possesses in a neighborhood of $(0,0)$ an analytic solution $z = W(u, v)$ satisfying $W_4(0,0) = W_5(0,0) = 0$; actually, $W_4(u, 0) = W_u(u, 0)$ and $W_5(u, 0) = W_v(u, 0)$ can be assigned arbitrarily (subject to the restriction that they be analytic, and small in absolute value).

The fact that $z = W$ is a solution of (10) can be written as

$$(12) \qquad \sum_{m=1}^{5} \sum_{k=1}^{5} b^0{}_{km} W_k W_m + d^0 = 0,$$

where $b^0{}_{km} = b_{km}(0,0)$, $d^0 = d(0,0)$. Put $z = W(u, v) + w$ in (8). This substitution transforms (8) into a partial differential equation for the unknown function w,

$$\sum_{m=1}^{5} \sum_{k=1}^{5} b_{km}(W_k + w_k)(W_m + w_m) + d = 0;$$

cf. (8 bis). Hence, from (12),

$$(13) \qquad \sum_{m=1}^{5} \Big(\sum_{k=1}^{5} b^0{}_{km} W_m \Big) w_k = P + \Pi,$$

where

$$(14) \qquad P = P(u, v) = \tfrac{1}{2} \sum_{k=1}^{5} \sum_{m=1}^{5} (b^0{}_{km} - b_{km}) W_m W_k + \tfrac{1}{2}(d^0 - d)$$

and

$$(15) \qquad \Pi \equiv \Pi(u, v, w_1, \cdots, w_5) = \sum_{m=1}^{5} \sum_{k=1}^{5} (b^0{}_{km} - b_{km}) W_m w_k - \tfrac{1}{2} \sum_{m=1}^{5} \sum_{k=1}^{5} b_{km} w_m w_k.$$

The expression on the left of (13) is of the form (6), where $a^k = \sum_{m=1}^{5} b^0{}_{km} W_m$ for $k = 1, \cdots, 5$, while $a_6 = 0$. Since (8) and (8 bis) are identical, and since $W_4(0,0) = W_5(0,0) = 0$, it is seen that

$$a^2a^2 - 4a^1a^3 = 4a^0d^0 \neq 0 \text{ for } (u, v) = (0, 0).$$

Consequently, Lemma 1 is applicable to equations of the form

$$\sum_{k=1}^{5} \left(\sum_{m=1}^{5} b^0{}_{km} W_m \right) w_k = f.$$

The proof of Lemma 2 will now be completed by the use of successive approximations.

5. The Successive Approximations. To this end, a few inequalities will now be collected.

If R is defined as in Lemma 1, let $0 < r < R$. Let the norm symbols $|\cdots|_\mu$, $\|\cdots\|_\mu$, where $0 < \mu < \lambda < 1$, refer, as above, to the circle $\mathfrak{C}_r$: $u^2 + v^2 < r^2$. Let $w(u, v)$, $^*w(u, v)$ denote two functions of class $C^2(\mu)$ in $\mathfrak{C}_r$ and let α be a number satisfying

$$(16) \qquad \| w \| \leqq \alpha \text{ and } \| {}^*w \| \leqq \alpha.$$

Choose $\beta = \beta(r)$ in such a way that

$$(17) \qquad | b_{km} - b^0{}_{km} |_\mu \leqq \beta \text{ and } | d - d^0 |_\mu \leqq \beta,$$

and that

$$(18) \qquad \beta = \beta(r) \to 0 \text{ as } r \to 0.$$

(that $\beta(r)$ can be chosen to satisfy (18), as well as (17), is a consequence of the assumption that b_{km} and d satisfy a Hölder condition of order $\lambda > \mu$). Finally, let

$$(19) \qquad \Pi(u, v) = \Pi(u, v, w_1, \cdots, w_5) \text{ and } {}^*\Pi(u, v) = \Pi(u, v, {}^*w_1, \cdots, {}^*w_5).$$

It follows from (14) and (15) that there exist constants A, B such that

$$(20) \qquad | P |_\lambda \leqq A\beta$$

and

$$(21) \qquad | \Pi |_\mu \leqq A\beta(\alpha + \beta) \text{ and } | \Pi - {}^*\Pi |_\mu \leqq B(\alpha + \beta) \| w - {}^*w \|.$$

The constants A, B can be chosen independent of r if $0 < r \leqq r_0 < R$ (for a fixed r_0).

If M denotes the constant occurring in Lemma 1, let $r > 0$ be so small that $1 - MA\beta(r) > 0$, and put

$$\tau = \tau(r) = MA\beta(1 + \beta)/(1 - MA\beta);$$

so that

$$(22) \qquad MA\beta + MA\beta(\tau + \beta) = \tau,$$

and $\tau(r) \to 0$ as $r \to 0$, by (18). If $r > 0$ is sufficiently small, then

$$q = q(r) = MB(\tau + \beta) < 1. \tag{23}$$

For sufficiently small $r > 0$, define a sequence of successive approximations as follows: $w^0(u, v) \equiv 0$ on $\mathcal{B}_r$ and, if $w^0, \cdots, w^j$ have been defined, $w^{j+1}(u, v)$ is that solution of

$$\sum_{k=1}^{5} \left(\sum_{m=1}^{5} b^0{}_{km} W_m \right) w^{j+1}{}_k = P(u, v) + \Pi(u, v, w^j{}_1, \cdots, w^j{}_5) \tag{24}$$

on $\mathcal{B}_r$ which is supplied by Lemma 1.

If $r > 0$ is so small that the above $\tau(r)$ is defined, then

$$\| w^j \| \leqq \tau \text{ for } j = 0, 1, \cdots. \tag{25}$$

In fact, the inequality (25) is trivial for $j = 0$. Suppose that (25) holds for some $j \geqq 0$. Since (20) and (21) are consequences of (16), it follows from (20) and the first inequality in (21) that

$$| P(u, v) + \Pi(u, v, w^j{}_1, \cdots, w^j{}_5) |_\mu \leqq A\beta + A\beta(\tau + \beta).$$

Then, from (7) and (22),

$$\| w^{j+1} \| \leqq MA\beta + MA\beta(\tau + \beta) = \tau.$$

This completes the induction proof of (25).

If $j \geqq 1$, equations (24) show that

$$\sum_{k=1}^{5} \left(\sum_{m=1}^{5} b^0{}_{km} W_m \right) (w^{j+1} - w^j)_k$$
$$= \Pi(u, v, w^j{}_1, \cdots, w^j{}_5) - \Pi(u, v, w^{j-1}{}_1, \cdots, w^{j-1}{}_5).$$

Since the solution w_f, supplied by Lemma 1, depends linearly on f, it follows from (7), and from the second inequality in (21), that

$$\| w^{j+1} - w^j \| \leqq BM(\tau + \beta) \| w^j - w^{j-1} \|.$$

Hence, if $r > 0$ is so small that (23) holds, then the series $\Sigma \| w^{j+1} - w^j \|$ is convergent. In view of the definition of $\| \cdots \|$, the series $\Sigma(w^{j+1} - w^j)$ is uniformly convergent, and its sum $w(u, v)$ is of class $C^2(\mu)$ on $\mathcal{B}_r$ and satisfies (13) on $\mathcal{B}_r$.

This completes the proof, since $z = W(u, v) + w(u, v)$ fulfills the assertion of Lemma 2.

Remark. It should be mentioned, for later reference, that $w = w(u, v; r)$ clearly is subject to the inequality $\| w \| \leqq (1 - q)^{-1}\tau$. Hence, $\| w \|$ can be made arbitrarily small by letting $r \to 0$.

6. The elliptic and parabolic cases when $n \geqq 3$. Let the functions $g_{ik} = g_{ik}(u, v)$ be of class $C^n(\lambda)$, where $n \geqq 2$, and let $\kappa(u, v) \neq 0$ in a vicinity $(0, 0)$. Following a formal device of Weingarten ([11], pp. 3-4; cf. Darboux [1], pp. 253-254), the problem will be reduced to the determination of a function $z = z(u, v)$ for which the matrix (γ_{ik}) is positive definite and $\kappa(u, v; \gamma_{ik}) \equiv 0$, where

$$\gamma_{11} = g_{11} - z_u^2, \qquad \gamma_{21} = \gamma_{12} = g_{12} - z_u z_v, \qquad \gamma_{22} = g_{22} - z_v^2.$$

Actually, $\kappa(u, v; \gamma_{ik})$ is not defined unless z is of class C^3. If it is assumed that $z = z(u, v)$ is of class C^3, the equation $\kappa(u, v; \gamma_{ik}) \equiv 0$ becomes a partial differential equation of second order, of the type (8), for z (so that no third order partial derivatives of z appear); cf. Darboux [1], p. 254. By a suitable normalization of (8), the function $a(u, v)$ becomes $-4(g_{11}g_{22} - g_{12}^2) \neq 0$, while $d(u, v)$ is identical with $4(g_{11}g_{22} - g_{12}^2)$. $\kappa(u, v; g_{ik}) \neq 0$. The functions $b_{km}(u, v)$ are quadratic polynomials in the g_{ik} and in their partial derivatives of first and second order.

If the g_{ik} are of class $C^n(\lambda)$, where $n \geqq 2$, then the coefficient functions of (8) are of class $C^{n-2}(\lambda)$. Hence, for sufficiently small r, there exist in $\mathfrak{B}_r$ solutions $z = z(u, v)$ of class $C^n(\mu)$. It follows from the Remark at the end of Section **5** that then the matrix (γ_{ik}) is positive definite, if r is sufficiently small. For, on the one hand, $z = W(u, v) + w(u, v)$ and $W_4(0, 0) = W_5(0, 0) = 0$, and, on the other hand, the function $w = w(u, v; r)$ has the property that $\| w \|$ can be made arbitrarily small by choosing r sufficiently small.

Assume that $n \geqq 3$. Then the corresponding functions γ_{ik} are of class $C^{n-1}(\mu)$, and so of class $C^2(\mu)$. An adaptation of the procedure applied in Section **3** shows that there exist functions $u^* = u^*(u, v)$, $v^* = v^*(u, v)$ of class $C^n(\mu)$ satisfying

$$du^{*2} + dv^{*2} = \gamma_{11} + 2\gamma_{12}\, du\, dv + \gamma_{22}\, dv^2.$$

Hence $x^1 = u^*(u, v)$, $x^2 = v^*(u, v)$, $x^3 = z(u, v)$ are of class $C^n(\mu)$ and satisfy (1). This completes the proof of the Theorem (Section **1**) if $n \geqq 3$ and $\kappa \neq 0$.

7. The case $\kappa \neq 0$ and $n = 2$. In order to complete the proof of the theorem for this case, considerations suggested by those of Weyl ([12], pp. 43-44; cf. van Kampen [3], p. 135) will be used.

For given functions g_{ik} of class C^2, the equation (2) and Green's theorem show that

$$(26)\qquad 2\iint_T \kappa\Delta\, du\, dv = -\tfrac{1}{2}\iint_T \Delta^{-3}D\, du\, dv$$
$$+\int_S \{\Delta^{-1}(g_{11v}-g_{12u})\,du + \Delta^{-1}(g_{12v}-g_{22u})\,dv\},$$

where S is any simple Jordan curve, of class C^1, and T is its interior. On the other hand, if the g_{ik} are only of class C^1, and if there exists a continuous function $\kappa(u, v)$ satisfying (26), then $\kappa(u, v)$ can be declared to be the curvature belonging to g_{ik}.

If the functions z and g_{ik} are of class C^2, it is easily seen that, in this sense, there exists a curvature $\kappa(u, v; \gamma_{ik})$, and that $\kappa = \phi(u, v, z_1, \cdots, z_5)/4(\det \gamma_{ik})^2$, where ϕ is the expression on the left of (8) as normalized in Section **6**. In fact, (2) shows that it is only necessary to establish the validity of the Green formula

$$-\int_S \delta^{-1}\{(z_v z_{uu} - z_u z_{uv})\,du + (z_v z_{uv} - z_u z_{vv})\,dv$$
$$= 2\iint_T \delta^{-1}(z_{uu}z_{vv} - z_{uv})^2 du\, dv$$
$$-\iint_T \delta^{-2}\{\delta_v(z_v z_{uu} - z_u z_{uv}) - \delta_u(z_v z_{uv} - z_u z_{vv})\}du\, dv,$$

where $\delta = (\det \gamma_{ik})^{\frac{1}{2}}$. Since this formula holds if $z(u, v)$ is of class C^3, it holds also if $z(u, v)$ is of class C^2, as can be seen by approximating z uniformly by polynomials in such a way that the first and second order partial derivatives also converge uniformly.

Let the g_{ik} be given functions of class $C^2(\lambda)$, and let z be of class $C^2(\mu)$ and a solution of (8). Then $\kappa(u, v; \gamma_{ik})$ is defined and is 0. Let $u = u(u', v')$, $v = v(u', v')$ be a transformation of class $C^3(\mu)$ satisfying $u(0, 0) = v(0, 0) = 0$ and $\partial(u, v)/\partial(u', v') \neq 0$, and having the property that the line-element belonging to

$$\gamma'_{km} = \sum_{j=1}^{2}\sum_{i=1}^{2} \gamma_{ij}(\partial u^i/\partial u'^k)(\partial u^j/\partial u'^m)$$

is conformal, that is, $\gamma'_{11} = \gamma'_{22} > 0$ and $\gamma'_{12} = 0$; cf. Section **3**. Let $\gamma = \gamma'_{11} = \gamma'_{22}$ and $\Gamma = \log \gamma$. The expressions D, Δ in (2), corresponding to conformal γ'_{ik}, are 0, γ, respectively. Since κ is invariant under transformations of class C^3, and since $\kappa \equiv 0$, it follows from (26) that

$$\int_S \{\partial\Gamma/\partial v'\, du' - \partial\Gamma/\partial u'\, dv'\} = 0$$

holds for any sufficiently smooth Jordan curve S. Hence Γ is analytic in a vicinity of $(u', v') = (0, 0)$. The proof can now be completed as in Section **6** (cf. Section **3**).

Remark. An inspection of the partial differential equation (8), occurring in Section **6**, shows that it contains second derivatives of the g_{ik} only in the particular combination which appeared in definition (2) of the curvature κ. Hence the above proof shows that the Theorem (Section **1**) can be refined as follows:

If $n \geqq 2$ and $0 < \mu < \lambda < 1$ and if the g_{ik} are of class $C^n(\lambda)$ and κ is of class $C^{n-1}(\lambda)$, then (1) *possesses solutions $x^j = x^j(u, v)$ of class $C^{n+1}(\mu)$.*

For the "parabolic case" of this theorem, cf. the end of Section **3**.

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A CLOSED SET OF ALGEBRAIC INTEGERS.*

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1. Introduction. A P. V. (Pisot-Vijayaraghavan) number is an algebraic integer, θ, all of whose conjugates over the rational field, with the exception of θ itself, lie inside the unit circle. P.V. numbers are of importance in certain problems of Diophantine approximation, chiefly because 0 and 1 are the only limit points of the sequence whose elements are the fractional parts of the powers of θ. Salem [5], [6], proved the surprising result that the set of P. V. numbers is closed, in the topological sense. Here we shall establish a similar theorem for a closely related class of algebraic integers.

Let us denote by S_1 the class of all P. V. numbers; let us further denote by S_2 the class of all algebraic integers, θ, which have one conjugate, θ' $(\neq \theta)$ lying, together with θ, outside the unit circle, while all other conjugates of θ lie inside the unit circle. The numbers of S_2 split naturally into two classes: S'_2, the class of real numbers in S_2 and S''_2, the class of all imaginary numbers in S_2. If θ is in S'_2, θ' is real, while if θ is in S''_2, θ' is imaginary and $\theta' = \bar{\theta}$. S_2, unlike S_1, is not closed, and neither is S'_2. In fact, the set of real quadratic algebraic integers is everywhere dense on the real line. Real quadratic integers outside the unit circle belong either to S_1 or S'_2. The integers of S_1 form a closed set; clearly the quadratic integers of S_1 cannot be everywhere dense. It follows that S'_2 is not closed and that also S_2 is not closed. Now S''_2 is not closed either, for S''_2 may have real limit points; however, the following theorem is true:

Theorem 1. *The imaginary limit points of S''_2 are in S''_2; the real limit points are in S_1. Thus $S_1 + S''_2$ is closed.*

The main part of this paper will be devoted to a proof of Theorem 1.

2. Proof of Theorem 1. Our proof is based upon a succession of lemmas.

In what follows, $c_0, c_1, \cdots, c_n, \cdots$ will be rational integers, while θ will be a complex number such that $|\theta| > 1$. We define the complex numbers $\rho_{1\,n}$ and $\rho_{2\,n}$ by the equations

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$$\rho_{1\,n} = \rho_{1\,n}(\theta) = c_n\theta - c_{n+1}, \qquad \rho_{2\,n} = \rho_{2\,n}(\theta) = c_n\theta\bar{\theta} - c_{n+1}(\theta + \bar{\theta}) + c_{n+2}.$$

We have the obvious identity

$$(1) \qquad \rho_{2\,n} = \rho_{1\,n}\bar{\theta} - \rho_{1\,n+1}.$$

LEMMA 1. *If $\rho_{2\,n}$ remains bounded as $n \to \infty$, then $c_n = o(|\theta| + \epsilon)^n$, where ϵ is an arbitrary positive number.*

Proof. Let $|\rho_{2\,n}| < M$, $n = 0, 1, 2, 3, \cdots$. From (1) we have $|\rho_{1\,n+1}| < |\rho_{1\,n}|\,|\theta| + M$, whence it easily follows, by induction on n, that

$$|\rho_{1\,n}| \leq |\rho_{1\,0}|\,|\theta|^n + M(|\theta|^n - 1)/|\theta| - 1 \leq A\,|\theta|^n,$$

where $A = |\rho_{1\,0}| + M/|\theta| - 1$. In other words, $|c_n\theta - c_{n+1}| \leq A\,|\theta|^n$, $n = 0, 1, 2, 3, \cdots$. Another induction on n gives $|c_n| \leq nA\,|\theta|^{n-1} + |c_0|\,|\theta|^n$, from which Lemma 1 follows immediately.

LEMMA 2. *If $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is a rational function and $\lim_{n\to\infty} a_n = 0$, then $g(z)$ has all its poles outside the unit circle.*

Proof. Let $1/\beta_1, 1/\beta_2, \cdots, 1/\beta_k$ be the poles of $g(z)$. Suppose that their orders are $r_1, r_2, \cdots, r_k$, respectively. For all n sufficiently large, $a_n = \sum_{i=1}^{\infty} P_i(n)\beta_i^n$, where $P_i(n)$ is a polynomial of degree $r_i - 1$. Assume that $g(z)$ has poles inside or on the unit circle. Let $1/\beta_1, \cdots 1/\beta_s$, be those of minimum modulus $1/p$, where $p \geq 1$. Since $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} \sum_{i=1}^{s} P_i(n)\beta_i^n = 0$. If $P_i(n) = \sum_{j=0}^{r_i-1} g_{ij}n^j$ and $\beta_i = p\phi_i$ where $|\phi_i| = 1$, then $\lim_{n\to\infty} p^n \sum_{i=1}^{s} P_i(n)\phi_i^n = \lim_{n\to\infty} p^n \sum_{j=0}^{r} n^j \sum_{i=1}^{s} g_{ij}\phi_i^n = 0$, where $r = \max_i (r_i - 1)$. It follows that $\lim_{n\to\infty} \sum_{i=1}^{s} g_{ir}\phi_i^n = \lim_{n\to\infty} \delta_n = 0$. Consider the system of linear equations

$$\sum_{i=1}^{s} g_{ir}\phi_i^{n+p} = \delta_{n+p}, \qquad p = 0, 1, \cdots, s-1.$$

This system has the solution

$$g_{ir} = \sum_{p=0}^{s-1} \lambda_{ip}\delta_{n+p}/\phi_i^n\Delta,$$

where the coefficients λ_{ip} are independent of n while Δ, the Vandermonde determinant derived from $\phi_1, \phi_2, \cdots, \phi_s$, is not zero. Letting $n \to \infty$, we

find that $g_{ir}=0$, $i=1,2,\cdots,s$. In the same way $g_{i,r-1}=0$, $g_{i,r-2}=0,\cdots$ $g_{i,0}=0$, $i=1,2,\cdots,s$, so that $P_i(n)\equiv 0$, $i=1,2,\cdots,s$.

LEMMA 3. *If* $\sum_{n=0}\rho_{2\,n}{}^2$ *converges and if infinitely many of the integers,* c_n, *are distinct from* 0, *then* (a) *if* $\theta\neq\bar{\theta}$, $\theta\,\varepsilon\,S''_2$, (b) *if* $\theta=\bar{\theta}$, $\theta\,\varepsilon\,S_1$.

Proof. We show first that $\{c_n\}$ is a recurring sequence, that is, there exist constants $a_1,\cdots,a_k$ such that

$$c_{n+k}+a_1c_{n+k-1}+\cdots+a_kc_n=0, \tag{2}$$

for all n greater than some fixed integer n_0. This will follow from the vanishing of the determinant,

$$D_n=\begin{vmatrix} c_0 & c_1 & \cdots & c_n\\ c_1 & c_2 & \cdots & c_{n+1}\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ c_n & c_{n+1} & \cdots & c_{2\,n}\end{vmatrix}=\begin{vmatrix} c_0 & c_1 & \rho_{2\,0} & \rho_{2\,1} & \cdots & \rho_{2\,n-2}\\ c_1 & c_2 & \rho_{2\,1} & \rho_{2\,2} & \cdots & \rho_{2\,n-1}\\ \cdot & \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & \cdot & & \cdot\\ c_n & c_{n+1} & \rho_{2\,n} & \rho_{2\,n+1} & \cdots & \rho_{2\,2n-2}\end{vmatrix},$$

for $n>n_0$. (Cf. [1]). From Hadamard's determinant theorem we have

$$D_n{}^2\leq(\sum_{j=0}^{n}c_j{}^2)(\sum_{j=1}^{n+1}c_j{}^2)(\sum_{j=0}^{n}\rho_{2j}{}^2)(\sum_{j=1}^{n+1}\rho_{2j}{}^2)\cdots(\sum_{j=n-2}^{2n-2}\rho_{2j}{}^2).$$

Since, by Lemma 1, $|c_j|<AH^j$, we have $\sum_{j=0}^{n}c_j{}^2\sum_{j=1}^{n+1}c_j{}^2<A_1H^{4n+6}$, where A_1 is a constant. Let $\sum_{j=h}^{\infty}\rho_{2j}{}^2=R_h$. Then $D_n{}^2\leq A_1H^{4n+6}R_0R_1R_2\cdots R_{n-2}$. Since $\lim_{h\to\infty}R_h=0$, $\lim_{n\to\infty}D_n=0$. But D_n is an integer. Hence there exists an integer, n_0, such that $D_n=0$ for all $n>n_0$, and $\{c_n\}$ is a recurring sequence.

Hence $f(z)=\sum_{n=0}^{\infty}c_nz^n$ is a rational function. Also, $g(z)=\sum_{n=0}^{\infty}\rho_{2\,n}(\theta)z^n$ is a rational function and $f(z)=z^2g(z)+c_0+(c_1-(\theta+\bar{\theta})c_0)z/(1-\theta z)(1-\bar{\theta}z)$. It follows from Lemma 2 that $g(z)$ has no poles inside the unit circle. $f(z)$ must have poles inside or on the boundary of the unit circle, since otherwise we should have $\lim_{n\to\infty}c_n=0$. These poles can be only at $1/\theta$ and $1/\bar{\theta}$. By the Fatou-Hurwitz theorem on rational functions whose power series have integral coefficients, cf. [3], the poles of $f(z)$ are the reciprocals of complete sets of conjugate algebraic integers. Thus $\theta\,\varepsilon\,S''_2$ if $\theta\neq\bar{\theta}$, while $\theta\,\varepsilon\,S_1$ if $\theta=\bar{\theta}$.

The argument used in obtaining Lemma 3 resembles that employed by Pisot in proving a somewhat similar result for S_1 (Cf. [2], [4]). The

obvious analogue of Lemma 3 for S_1 is also true; if $\sum_{n=0}^{\infty} \rho_{1\,n}{}^2$ converges and if infinitely many of the integers c_n are distinct from 0, then $\theta \varepsilon S_1$.

LEMMA 4. *If $\theta \varepsilon S''_2$ and k is a positive integer, there exist rational integers $c_0, c_1, \cdots, c_n, \cdots$, independent of k, such that*

$$\sum_{n=0}^{k-1} c_n{}^2 + \sum_{n=k}^{2k-1} (c_n - c_{n-k}(\theta^k + \bar{\theta}^k))^2 + \sum_{n=2k}^{\infty} (c_n - c_{n-k}(\theta^k + \bar{\theta}^k) + c_{n-2k}\theta^k\bar{\theta}^k)^2 \tag{3}$$
$$= 1 + (\theta^k + \bar{\theta}^k)^2 + (\theta^k\bar{\theta}^k)^2.$$

Proof. Let θ be a root of the primary irreducible polynomial $P(z)$ which has rational integral coefficients. Let $P(z)$ have degree r and let $Q(z) = z^r P(1/z)$. In some neighborhood of the origin we have $P/Q = \sum_{n=0}^{\infty} c_n z^n$, where the coefficients c_n are rational integers, since $Q(0) = 1$. The function P/Q is regular for $|z| \leqq 1$, save possibly for poles at $1/\theta$ and $1/\bar{\theta}$; and the function $(1 - \theta^k z^k)(1 - \bar{\theta}^k z^k)P/Q$ is regular at all points of $|z| \leq 1$. Setting $z = e^{i\phi}$, we obtain from Parseval's equality

$$\frac{1}{2\pi}\int_0^{2\pi} |(1-\theta^k z^k)(1-\bar{\theta}^k z^k)P/Q|^2\, d\phi = \sum_{n=0}^{k-1} c_n{}^2 + \sum_{n=k}^{2k-1} (c_n - c_{n-k}(\theta^k + \bar{\theta}^k))^2$$
$$+ \sum_{n=2k}^{\infty} (c_n - c_{n-k}(\theta^k + \bar{\theta}^k) + c_{n-2k}\theta^k\bar{\theta}^k)^2$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} |(1-\theta^k z^k)(1-\bar{\theta}^k z^k)|^2\, d\phi = 1 + (\theta^k + \bar{\theta}^k)^2 + (\theta^k\bar{\theta}^k)^2.$$

Since $|P/Q| = 1$ when $|z| = 1$, these two expressions are equal. Lemma 4 is an extension of a lemma proved by Siegel ([7]) for S_1 and $k = 1$. The same argument is used here. We remark that the function P/Q will actually have poles at $1/\theta$ and $1/\bar{\theta}$ unless $P(1/\theta) = P(1/\bar{\theta}) = 0$. Since θ is in S''_2, this can occur only if θ is a biquadratic unity.

LEMMA 5. *Let $c_0, c_1, \cdots, c_n, \cdots$ be chosen as in Lemma 4. Then, if θ is not a biquadratic unity, c_k and c_{2k} are not both zero, $k = 0, 1, 2, \cdots$.*

Proof. Since $c_0 = P(0)/Q(0) = P(0)$, and $P(z)$ is irreducible, $c_0{}^2 \geq 1$. Suppose that $c_k = c_{2k} = 0$, for some particular value of $k > 0$. Then the terms on the left-hand side of (3) corresponding to $n = 0$, $n = k$ and $n = 2k$ have the sum $c_0{}^2(1 + (\theta^k + \bar{\theta}^k)^2 + (\theta^k\bar{\theta}^k)^2)$. This is compatible with Lemma 4 only if $c_0 = \pm 1$, and all remaining coefficients are 0, so that P/Q is regular in $|z| \leq 1$ and θ is a biquadratic unity.

LEMMA 6. *Let* $c_0, c_1, \cdots, c_n, \cdots$, *be chosen as in Lemma* 4. *Then*

$$|c_m| \leq (1 + (\theta^{m+1} + \bar{\theta}^{m+1})^2 + (\theta^{m+1}\bar{\theta}^{m+1})^2)^{1/2} < |\theta|^{2m+2} + 2, \qquad m = 0, 1, 2, \cdots.$$

Proof. Set $k = m + 1$ in Lemma 4 and observe that all terms on the left-hand side of (3) are non-negative.

We turn now to the proof of Theorem 1. Let α be the limit of a sequence $\{\theta_s\}$ of algebraic integers in S''_2. We are to show that α is either in S''_2 or in S_1.

Proof. The sequence $\{\theta_s\}$ is bounded; for all s we have $|\theta_s| < M$. Hence the sequence $\{\theta_s\}$ can contain only a finite number of biquadratic unities, for the number of algebraic integers of a given degree, lying, together with their conjugates in a bounded region, is finite. We may therefore suppose that these biquadratic unities are deleted from $\{\theta_s\}$.

Corresponding to each element, θ_s, of our sequence, we construct, as in Lemma 4, a sequence of integers c_{ns} so that

$$(4) \qquad c_{0\,s}^2 + (c_{0\,s}(\theta_s + \bar{\theta}_s) - c_{1\,s})^2 + \sum_{n=0}^{\infty} \rho_{2\,n}^2(\theta_s) = 1 + (\theta_s + \bar{\theta}_s)^2 + (\theta_s\bar{\theta}_s)^2.$$

From Lemma 6 we have

$$(5) \qquad |c_{n\,s}| < M^{2n+2} + 2, \qquad n = 0, 1, 2, \cdots, s = 1, 2, 3, \cdots.$$

The bounded sequence $c_{0\,s}$ consists only of integers; hence some integers must occur infinitely often. Call one of these integers c_0. We may extract an infinite subsequence $\{\theta_{s_0}\}$ from our original sequence such that $c_{0\,s_0} = c_0$, $s_0 = 1, 2, 3, \cdots$. Clearly $c_0 \neq 0$.

Similarly, the bounded sequences $c_{1\,s_0}$ and $c_{2\,s_0}$, consist only of integers, so that some integers must occur infinitely often in each of these sequences. Choose one of these integers from each sequence and call them c_1 and c_2. We infer from Lemma 5 that one of the sequences $c_{1\,s_0}$ and $c_{2\,s_0}$ contains infinitely many elements distinct from 0, hence at least one of the integers c_1, c_2, may be chosen distinct from 0. We may extract an infinite subsequence $\{\theta_{s_1}\}$ from the sequence $\{\theta_{s_0}\}$ such that $c_{0\,s_1} = c_0$, $c_{1\,s_1} = c_1$, $c_{2\,s_1} = c_2$, $s_1 = 1, 2, 3, \cdots$.

Proceeding in this way, we construct an infinite sequence of integers: $\{c_n\}$. Repeated application of Lemma 5 enables us to choose the integers c_q and c_{2q} (where $2q$ contains 2 to an odd power) in such a way that one of them is distinct from 0. Moreover, for any integer, $m \geq 0$, there exists an infinite subsequence $\{\theta_{s_m}\}$ of our original sequence $\{\theta_s\}$ such that $c_{n\,s_m} = c_n$, $n = 0, 1, 2, \cdots, m$, $s_m = 1, 2, 3, \cdots$.

We now demonstrate the convergence of the series $\sum_{n=0}^{\infty} \rho_{2\,n}^2(\alpha)$. Theorem 1 will then follow from Lemma 3, provided that $|\alpha| \neq 1$. We have

$$\sum_{n=0}^{m-2} \rho_{2\,n}^2(\alpha) = \sum_{n=0}^{m-2} [\rho_{2\,n}^2(\alpha) - \rho_{2\,n}^2(\theta_{s_m})] + \sum_{n=0}^{m-2} \rho_{2\,n}^2(\theta_{s_m}). \tag{6}$$

Let ϵ be an arbitrary positive number. Choose s_m so large that $|\alpha - \theta_{s_m}| < \epsilon$. Observing that $c_n = c_{n\,s_m}$, $n = 0, 1, 2, \cdots, m$, we find that

$$|\rho_{2\,n}(\alpha) - \rho_{2\,n}(\theta_{s_m})| \leq |c_n| [|\alpha|^2 - |\theta_{s_m}|^2] + 2|c_{n+1}| |\alpha - \theta_{s_m}|,$$
$$n = 0, 1, \cdots, m-2.$$

From (5) we obtain $|c_n| = |c_{n\,s_m}| < M^{2m+2} + 2$, $n = 0, 1, 2, \cdots, m$, whence

$$|\rho_{2\,n}(\alpha) - \rho_{2\,n}(\theta_{s_m})| < (M^{2m+2} + 2)(2M + 2)\epsilon, \tag{7}$$
$$n = 0, 1, 2, \cdots, m-2.$$

Similarly

$$|\rho_{2\,n}(\alpha) + \rho_{2\,n}(\theta_{s_m})| < 2(M^{2m+2} + 2)(M + 1)^2, \tag{8}$$
$$n = 0, 1, 2, \cdots, m-2.$$

Multiplying (8) and (7) and summing over n, we have

$$\left|\sum_{n=0}^{m-2} [\rho_{2\,n}^2(\alpha) - \rho_{2\,n}^2(\theta_{s_m})]\right| < 4(M^{2m+2} + 2)^2(M + 1)^3(m-1)\epsilon,$$
$$n = 0, 1, 2, \cdots, m-2.$$

Since ϵ is arbitrary, we can conclude from (6) and (4) that $\sum_{n=0}^{m-2} \rho_{2\,n}^2(\alpha) \leq 1 + 4M^2 + M^4$. Consequently, $\sum_{n=0}^{\infty} \rho_{2\,n}^2(\alpha)$ is convergent.

We complete the proof by showing that the excluded case $|\alpha| = 1$ cannot arise.

Any positive integral power of a number in S''_2 is either in S''_2 or in S_1. If the sequence $\{\theta_s\}$ converges to $\alpha = e^{i\phi}$, then the set of all positive integral powers of all numbers in the sequence will have a limit point, α_R, on each circle $r = R$, $R \geq 1$. If $R > 1$, then α_R must, by what we have already proved, belong to S''_2 or S_1. Every number in S''_2 or S_1 is algebraic and must have an algebraic modulus. But R may be transcendental. This contradiction shows that $|\alpha| \neq 1$.

3. Concluding remarks. We could have dispensed with Lemma 1 altogether in our proof of Theorem 1 if we had strengthened the hypothesis

in Lemma 3 by requiring that $c_n = O(B^n)$, for some $B > 1$. This condition is fulfilled by the integers c_n used in the proof of Theorem 1, on account of Lemma 6. However, we feel that Lemma 3 is of some interest in its own right, and we have stated it in as strong a form as possible.

Either of Salem's proofs that S_1 is closed may be employed, with more or less obvious modifications, to prove that part of Theorem 1 which concerns the imaginary limit points of S''_2; however, we were unsuccessful in our attempts to adapt them to the rest of Theorem 1. The basic idea underlying our proof is similar to that underlying his first proof [5], but there are considerable differences in detail. Our Lemma 3 plays the rôle of Pisot's theorem, while Lemma 4 play the rôle of Salem's lemma.

We have tried to extend Theorem 1 to still wider classes of algebraic integers. Let S_k be the class of all algebraic integers, θ, having exactly k conjugates, including θ itself, outside the unit circle, and all remaining conjugates inside the unit circle, with none *on* the circle. An argument just like that given in the introduction for $k = 2$ shows that S_k is not closed if $k > 1$. If, ignorant of this fact, we were to try to use the methods of this paper to prove that S_k is closed, we should find that, although our sequence $\{\theta_s\}$ is bounded, the conjugates $\theta_s^{(i)}$, $i = 1, 2, \cdots, k-1, s = 1, 2, \cdots$ may very well be unbounded. It is natural then, to impose the further restriction that the conjugates be bounded also. The right-hand side of the analogue of Lemma 4 with θ replaced by θ_s, is then a bounded function of s, a property which we used several times in our proof of Theorem 1. We can then extract a subsequence of the sequence $\{\theta_s\}$ such that we can form $k-1$ sequences of conjugates which are also convergent with limits, say, $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$. One might expect that then each of the numbers $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ would belong to some S_r with $r \leq k$. However, one finds that the complete analogue of Lemma 3 is false. All that we can say at the present time is that at least one of the numbers $\alpha, \alpha_1, \cdots, \alpha_{k-1}$ belongs to some S_r, with $r \leq k$.

Since $S_1 + S''_2$ is closed, every point of $S_1 + S''_2$ is either a limit point or an isolated point. It would be of interest to determine which points of $S_1 + S''_2$ are limit points and which are isolated points. Siegel ([7]) has carried through a similar investigation for S_1; he has shown that S_1 has no limit points in the closed interval, $[1, 2^{\frac{1}{2}}]$ and has found all the isolated points in this interval. However, nothing is known beyond this; the identity of the smallest limit point has not yet been determined. The problem for $S_1 + S''_2$ appears to be even more difficult. We can merely state that, as shown at the conclusion of the proof of Theorem 1, $S_1 + S''_2$ has no limit points on the unit circle.

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DYNAMICAL SYSTEMS WITH INDETERMINACY.*[1]

By Wilfred Kaplan.

1. Introduction.

1.1. Classical dynamical systems. The classical theory of dynamical systems concerns the properties of the solutions of differential equations

$$dx_i/dt = f_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

(in local coordinates) in an n-dimensional manifold M, the phase space. This theory has been developed to a considerable extent through the efforts of Poincaré, G. D. Birkhoff and others. The results obtained concern methods of obtaining solutions (usually in series form), existence and stability of periodic solutions, nature of singular points, existence of wandering motions, recurrent motions and other special types of solutions, and ergodic theory.

In general one is impressed with the complexity of the problem. For $n = 2$ a complete analysis of all possibilities appears achievable, but for $n = 3$ or more such a program appears hopeless.

1.2. Introduction of indeterminacy. In the present paper the statement of the problem is modified as follows. Instead of studying the exact solutions of the above differential equations, one considers the ϵ-solutions, solutions $x_i = x_i(t)$ for which the velocity vector dx_i/dt differs from the prescribed vector f_i by 'less than ϵ' (a condition to be made precise below). In effect one thus replaces the given direction at each point by a 'cone of directions,' so that the direction of the solution at each point has a certain degree of indeterminacy.

Such a modification is not unfamiliar, a similar approach being used in the theory of random processes. Here, however, the probability aspect will not be emphasized, the answers to all questions being given in the form of "yes" or "no" (possibility or impossibility).

The reasons for the introduction of indeterminacy are as follows. It is felt that this is a more realistic formulation of the physical problem for

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which the theory of dynamical systems has been devised. In every concrete physical problem a certain amount of approximation is needed before the problem can be reduced to precise mathematical form. In many cases the approximations used can be estimated, at least roughly, so that one has some knowledge concerning the "ϵ" involved. Thus there appears to be a natural physical basis for the present approach. But, as will be seen below, the introduction of indeterminacy has the further advantage of greatly simplifying the mathematical description of the family of solutions.

1.3. Summary of results. One introduces a partial order in M by the definition: $x < y$ if there is an ϵ-solution from x to y with increasing t. It may then happen that $x < y < x$ for some x and y, so that there is a periodic ϵ-solution through x and y. In such a case, or if x coincides with y, one writes: $x \equiv y$, and thereby defines an equivalence relation which decomposes M into equivalence classes, termed "states." Each state consists either of one element, in which case the state is termed *transitory*, or of the points of an open subset of M, in which case the state is termed *stationary*. The state containing x is denoted by $\pi(x)$, the set of all y such that $y < x$ is denoted by $\alpha(x)$. A stationary state $\pi(x)$ is called *ω-stable* if $x < y$ implies $y \in \pi(x)$, *α-stable* if $y < x$ implies $y \in \pi(x)$. It is shown that, if M is compact, there are in all a finite number of ω-stable stationary states $\pi(z_1), \cdots, \pi(z_N)$ and every x in M satisfies an inequality: $x < z_j$ for at least one z_j; a similar statement applies with regard to α-stability. The sets $F_j = \alpha(z_j)$ are open and cover M. The nerve of this covering is a complex Φ which describes the qualitative structure of the family of ϵ-solutions. It is shown that the complex Φ is 'in general' insensitive to small changes in the allowed error ϵ. It is further shown that, except for certain critical choices of ϵ, the complex Φ can be computed from an analysis of the given vector field at a finite number of points of M. This last result is the basis of a numerical method for obtaining the ω-stable stationary states and the complex Φ. This is illustrated in the case of the Van der Pol equation.

2. Flows in partially ordered sets.

2.1. Partial order and flow. Let M be a set of elements: $x, y, \cdots$ with order relation $<$ such that $x < y$ and $y < z$ imply $x < z$. Then M will be termed *partially ordered*. The set of all y such that $y < x$ will be denoted by $\alpha(x)$; the set of all y such that $x < y$ will be denoted by $\omega(x)$. If, for each x in M, the set $\alpha(x)$ is non-void, then the partial order will be termed α-extensive; if, for each x in M, the set $\omega(x)$ is non-void, then the partial order

will be termed *ω-extensive*; if both $\alpha(x)$ and $\omega(x)$ are non-void for all x in M, then the order will be termed *extensive*. If the relation $x < x$ holds for no x in M, then the partial order will be termed *proper*.

The partial order can be thought of as representing a flow with indeterminacy in M. Thus the relation $x < y$ can be interpreted as meaning: it is possible to move from x to y with increasing time t. In a classical dynamical system the relation $x < y$ would mean simply: the trajectory $z = \phi(t)$ passing through x when $t = 0$ passes through y for some positive t; here the indeterminacy is degenerate: if $x < y$, then it is not merely possible but inevitable that the trajectory through x reaches y at a later time. The indeterminacy can arise, as will be discussed in Section **3** below, through replacing the exact differential equations of a classical dynamical system by inequalities, reflecting incomplete knowledge of the system.

A relation: $x < x$ is simply a statement that there is a closed (periodic) orbit through x. There is thus no reason for excluding this possibility for a general flow. Accordingly, the partial orders consider here are in general *improper*.

2.2. The states $\pi(x)$. Let M denote a fixed partially ordered set. For each x in M the set $\pi(x)$ is defined by the equation

$$\pi(x) = \{x\} \cup (\alpha(x) \wedge \omega(x)).$$

Thus $\pi(x)$ consists of x plus all y such that $x < y$ and $y < x$, hence of x plus all elements on closed trajectories through x.

If $\alpha(x) \wedge \omega(x) = 0$, so that $\pi(x)$ reduces to $\{x\}$, then the element x will be termed *transitory*. If $\alpha(x) \wedge \omega(x) \neq 0$, then x will be called *stationary*. If x is stationary, then $x \, \varepsilon \, \alpha(x) \wedge \omega(x)$. For $y \, \varepsilon \, \alpha(x) \wedge \omega(x)$ implies $x < y$ and $y < x$, so that $x < x$. Hence in this case $\pi(x) = \alpha(x) \wedge \omega(x) \supseteqq \{x\}$. The same reasoning shows that, if x is stationary, then every phase y in $\pi(x)$ is stationary.

Each set $\pi(x)$ will be called a *state*, and will be termed a *transitory state* or a *stationary state* according to whether it consists of one transitory element x or of one or more stationary elements $x, y, \cdots$.

2.3. Ordering of the states. If $\pi(x) \wedge \pi(y) \neq 0$, then $\pi(x) = \pi(y)$. For if $z \, \varepsilon \, \pi(x)$ and $w \, \varepsilon \, \pi(x) \wedge \pi(y)$, then $z \leqq x \leqq w \leqq y$ and $y \leqq w \leqq x \leqq z$, so that $z \, \varepsilon \, \pi(y)$. Thus $\pi(x) \subseteqq \pi(y)$ and similarly $\pi(y) \subseteqq \pi(x)$, so that $\pi(x) = \pi(y)$.

It follows that the states $\pi(x)$ form a decomposition of M into disjoint

subsets. The notation $x \equiv y$ will be used for the corresponding equivalence relation: i. e., $x \equiv y$ if $\pi(x) = \pi(y)$.

If $x \equiv y$, $z \equiv w$, and $x < z$, then $y < w$. For $y \leqq x < z \leqq w$. Hence one can define: $\pi(x) < \pi(y)$ if $\pi(x) \neq \pi(y)$ and $z < w$ for some $z \equiv x$ and some $w \equiv y$. This ordering satisfies the transitive law, so that the states are now partially ordered. This ordering is proper, for $\pi(x) < \pi(x)$ is ruled out by definition.

A stationary state $\pi(x)$ will be called *ω-stable* if $\pi(x)$ is maximal, i. e., if there is no $\pi(y)$ such that $\pi(y) > \pi(x)$. Similarly, $\pi(x)$ is *α-stable* if $\pi(x)$ is minimal.

The given partially ordered set M will be said to have *finite ω-stability type* if the following two conditions hold:

(a) the number of ω-stable states is finite;

(b) for every state $\pi(x)$ there exists an ω-stable state $\pi(z)$ such that $\pi(x) \leqq \pi(z)$.

The notion of *finite α-stability type* is defined in analogous fashion.

2. 4. Stability complexes. Let M have finite ω-stability type and let $\pi(x_1), \cdots, \pi(x_N)$ be the ω-stable states. By condition (b) the sets $F_1 = \alpha(x_1), \cdots, F_N = \alpha(x_N)$ cover M. The nerve of the covering is a finite simplicial complex Φ, the *ω-stability complex*. Thus the N vertices of Φ are the sets $F_1, \cdots, F_N$, the edges are those pairs (F_j, F_k) for which $j \neq k$ and $F_j \wedge F_k \neq 0$, the 2-simplexes are those triples (F_h, F_j, F_k) for which $h \neq j$, $j \neq k$, $k \neq h$ and $F_h \wedge F_j \wedge F_k \neq 0$, etc. In the same way, if M has finite α-stability type, the covering of M by sets G_k forms a finite complex Γ, the *α-stability complex*.

Each simplex of Φ corresponds to a subset of M with given uncertainty about the ultimate future. Thus the 2-simplex (F_1, F_2, F_3) corresponds to the subset $F_1 \wedge F_2 \wedge F_3$ of those elements from which the states $\pi(x_1)$, $\pi(x_2)$, $\pi(x_3)$ are possible future states. In dynamical language, these elements are the initial conditions from which the ω-stable states $\pi(x_1)$, $\pi(x_2)$, $\pi(x_3)$ can be reached. Similar remarks apply to Γ with regard to the past.

3. Dynamical systems with indeterminacy.

3. 1. The phase space M. It will be assumed that M is a compact n-dimensional manifold of class $C^{(3)}$. It will further be assumed that M has a Riemannian metric: $ds^2 = a_{ij}dx^i dx^j$ in terms of local coordinates

$(x^1,\cdots,x^n)$, where the a_{ij} are of class C'. Hence M becomes a metric space with metric $\rho(x,y)$ equal to the g. l. b. of the lengths of all paths from x to y in M. Finally, it will be assumed that a fixed *dynamical system* is given in M: i.e., a vector field $v=v(x)$ whose contravariant components $v^i(x^1,\cdots,x^n)$ are of class C'. The differential equations

$$(1)\qquad dx^i/dt = v^i(x^1,\cdots,x^n) \qquad (i=1,\cdots,n)$$

then determine the trajectories of the given dynamical system. The points of M will be referred to as *phases*; M will be termed the *phase space*.

Associated with M is the fibre space V of all vectors u in M. If local coordinates $(x^1,\cdots,x^n)$ are chosen in M, then local coordinates $(x^1,\cdots,x^n;u^1,\cdots,u^n)$ are determined in V; for this reason the points of V will be denoted by $(x;u)$. The vectors u at each point form a normed linear space with norm $\|u\|$ equal to $(a_{ij}u^iu^j)^{\frac{1}{2}}$. The subset of V of all vectors with non-zero norm will be denoted by V'.

The space V, with the above coordinate systems, is a differentiable manifold and, as a normal separable space, is metrisable. The metric will be denoted by $\sigma((x;u),(y;w))$.

3.2. Indeterminacy functions. By an *indeterminacy function* in M will be meant a real-valued function $\epsilon(x;u)$ defined in V and continuous in V' and satisfying the conditions:

$$(2)\qquad \epsilon(x;u)>0;\quad \epsilon(x;u)=\epsilon(x;ku),\ k>0;\quad \epsilon(x;0)=1.$$

Thus ϵ is a function of direction only. For many purposes it will be convenient to use the related function $E(x;u)$ defined by the equation:

$$(3)\qquad E(x;u)=\|u\|/\epsilon(x;u).$$

Thus $E(x;u)$ is continuous in V and $E(x;ku)=kE(x;u)$ for $k>0$.

The indeterminacy function $\epsilon(x;u)$ will be said to be *convex* if the set of all vectors u at x such that $\|u\|<\epsilon(x;u)$ is convex. This is equivalent to the condition that the set of u such that $E(x;u)<1$ be convex or, because of the homogeneity of E, to the condition that

$$(4)\qquad E(x;\lambda u_1+(1-\lambda)u_2)\leqq \lambda E(x;u_1)+(1-\lambda)E(x;u_2),\quad 0\leqq\lambda\leqq 1.$$

Corresponding to the indeterminacy function $\epsilon(x;u)$ there is an *indeterminacy neighborhood* $U(x;v;\epsilon)$ of each vector v (at x) of the given dynamical system: namely, the set of all vectors w at x that $E(x;w-v)<1$.

By an *allowed solution relative to* $\epsilon(x;u)$ or, more briefly, by an *ϵ-solution*, will be meant a path $x=x(t)$, defined over some t-interval (perhaps infinite),

which is piecewise smooth and is such that at each point x of the path the vector dx/dt satisfies:

$$\| dx/dt - v(x) \| < \epsilon(x; dx/dt - v(x)) \tag{5}$$

(for either choice of dx/dt at corners) or, in terms of the E-function:

$$E(x; dx/dt - v(x)) < 1. \tag{5'}$$

Thus the velocity vector dx/dt is required to lie within the indeterminacy neighborhood $U(x; v(x); \epsilon)$ at each x.

The classical existence theorem for differential equations guarantees existence of a unique solution of (1) through each phase x. Here one can state that *through each phase x_0 and for each vector w_0 in $U(x_0; v(x_0); \epsilon)$ there is an ϵ-solution (not unique) $x = x(t)$ for which the velocity vector at x_0 is w_0.* For, in local coordinates, the straight line $x^i = x_0{}^i + w_0{}^i t$, $|t| < t_1$, will be such a solution, for t_1 sufficiently small. This follows from the fact that $E(x_0 + w_0 t; w_0 - v(x_0 + w_0 t))$ is continuous in t at $t = 0$, is < 1 for $t = 0$ and hence remains < 1 for $|t|$ sufficiently small. An infinity of such solutions can clearly be provided. In general, all the ϵ-solutions through x_0 form a configuration like a spray from a nozzle.

3.3. Partial order in M. A partial order in M is now introduced by the definition: $x < y$ if there is an ϵ-solution passing through x and y for values t_1 and t_2 respectively of t, with $t_1 < t_2$. This clearly obeys the transitive law, while the possibility of periodic solutions makes the order in general improper. The ordering is an extensive one, by virtue of the existence theorem proved in **3.2**. The dependence of the order on the choice of $\epsilon(x; u)$ will be indicated, where necessary, by the notation: $x < y$ [ϵ]. For the present a fixed choice of $\epsilon(x; u)$ will be assumed.

THEOREM 1. *For each x_0 in M, the sets $\alpha(x_0)$ and $\omega(x_0)$ are open.*

Proof. Let $y \in \omega(x_0)$, i.e., $x_0 < y$, so that there is an ϵ-solution $x = x(t)$, $t_1 \leqq t \leqq t_2$, with $x(t_1) = x_0$, $x(t_2) = y$, $t_1 < t_2$. Choose local coordinates at y and let t_3 be chosen so that $t_1 < t_3 < t_2$ and so that $x(t)$ lies in the chosen neighborhod of y for $t_3 \leqq t \leqq t_2$. For each constant vector u such that $u^i u^i = 1$ and for λ_1 sufficiently small, the paths

$$x^i = x^i(t) + \lambda u^i (t - t_3), \qquad t_3 \leqq t \leqq t_2, \qquad 0 \leqq \lambda < \lambda_1,$$

then lie within the chosen neighborhood of y and are furthermore ϵ-solutions. This follows at once, since

$$E(x(t)+\lambda u(t-t_3)\,;\,dx/dt+\lambda u-v(x+\lambda u(t-t_3)))$$

is continuous in t and λ and is less than 1 for $\lambda = 0$. The paths all pass through $z = x(t_3)$ for $t = t_3$, and for $t = t_2$ fill out a spherical neighborhood of radius $\lambda_1(t_2 - t_3)$ about y. Each phase x of this spherical neighborhood satisfies $z < x$ and, by the transitive law, $x_0 < x$. Hence $\omega(x_0)$ is open and, in the same way, $\alpha(x_0)$ is open.

COROLLARY. *If x is a stationary phase, then $\pi(x)$ is open.*

For here $\pi(x) = \alpha(x) \triangle \omega(x)$.

THEOREM 2. *M, as partially ordered above, has finite ω-stability type and finite α-stability type.*

Proof. This is an immediate consequence of compactness. For each x in M one can choose $y > x$, since M is extensively ordered. Hence all the sets $\alpha(x)$ together cover M. Hence there exist a finite number of these: $\alpha(x_1),\cdots,\alpha(x_N)$ which together cover M. By removal of some of these sets, if necessary, one can reduce this covering to a *minimal* one, i. e., one such that removal of any one set would destroy the property of covering M. It will be assumed that this has been done, so that $(\alpha(x_1),\cdots,\alpha(x_N))$ is a minimal covering. It now follows that each x_j is stationary and that each $\pi(x_j)$ is ω-stable, for $j = 1,\cdots,N$. For let $y \,\varepsilon\, \omega(x_j)$, $y \neq x_j$. Then $y < x_k$ for some k and hence $x_j < x_k$. This implies that $j = k$, since the covering is minimal, and hence $x_j < x_j$, so that x_j is stationary. The same reasoning shows that $y > x_j$ implies that $y \equiv x_j$, i. e., that $\pi(x_j)$ is ω-stable. There can be no further ω-stable states, since the $\alpha(x_j)$ form a covering. Hence there are a finite number, N, of ω-stable states $\pi(x_j)$ $(\mathrm{j} = 1,\cdots,N)$ and the sets $F_j = \alpha(x_j)$ cover M. In the same manner, one concludes that there are a finite number of α-stable stationary states $\pi(y_k)$ $(k = 1,\cdots,P)$ and the sets $G_k = \omega(y_k)$ cover M. Thus M has finite stability types.

One can accordingly define the complex $\Phi = \Phi(\epsilon)$ as the nerve of the covering by the F_j and the complex $\Gamma = \Gamma(\epsilon)$ as the nerve of the covering by the G_k. The following theory will be given for Φ alone, the theory for Γ being parallel.

4. VARIATION OF THE INDETERMINACY FUNCTION.

4. 1. Ordering of indeterminacy functions. The fibre space V^* of unit vectors in M is a compact space, and one can define the norm $\| f \|$ of a continuous real-valued function in V^* as l. u. b. $| f |$ over V^*. One can

define the distance between two such functions f_1, f_2 as $\| f_1 - f_2 \|$, and thereby obtains a metric space. Because of (2), the indeterminacy functions $\epsilon(x; u)$ depend on direction only, hence are determined completely by their values on V^*. The norm $\| \epsilon \|$ of $\epsilon(x; u)$ will mean the norm of $\epsilon(x; u)$ restricted to V^*, and the distance $\| \epsilon_1 - \epsilon_2 \|$ is defined in the same way. It follows that the indeterminacy functions also form a metric space, denoted by $\mathcal{E}$. This space will be partially ordered by the definition: $\epsilon_1 \ll \epsilon_2$ if $\epsilon_1(x; u) < \epsilon_2(x; u)$ for each element $(x; u)$ of V'.

For M itself one then concludes that, if $x < y\ [\epsilon_1]$ and $\epsilon_1 \ll \epsilon_2$, then $x < y\ [\epsilon_2]$.

The complex Φ above is determined by a covering of M. For brevity, Φ will also be identified with the covering. Thus the statement that Φ_1 is a refinement of Φ_2 means that each set (vertex) of Φ_1 is a subset of a set of the covering Φ_2. It should be noted that the covering Φ for given ϵ is a minimal one; this follows from the proof of Theorem 2, or from the definition of finite ω-stability type.

THEOREM 3. *Let ϵ_1 and ϵ_2 be indeterminacy functions and let $\Phi_1 = \Phi(\epsilon_1)$ and $\Phi_2 = \Phi(\epsilon_2)$ be the corresponding ω-stability complexes. If $\epsilon_1 \ll \epsilon_2$, then Φ_1 is a refinement of Φ_2.*

Proof. Let $\pi(x_1), \cdots, \pi(x_{N_1})$ be the ω-stable stationary states relative to ϵ_1; let $\pi(y_1), \cdots, \pi(y_{N_2})$ be the ω-stable states relative to ϵ_2. Let $F_{1j} = \alpha(x_j)$ (relative to ϵ_1), let $F_{2k} = \alpha(y_k)$ (relative to ϵ_2) $(j = 1, \cdots, N_1, k = 1, \cdots, N_2)$. Then for each x_j there is some y_k such that $x_j \leqq y_k\ [\epsilon_2]$. For each x in F_{1j} one has $x < x_j\ [\epsilon_1]$, hence, since $\epsilon_1 \ll \epsilon_2$, $x < x_j\ [\epsilon_2]$, and hence $x < y_k\ [\epsilon_2]$. Thus $F_{1j} \subseteq F_{2k}$. Hence Φ_1 is a refinement of Φ_2.

4.2. Critical and non-critical indeterminacy functions. *Definition.* The covering Φ_1 of M by sets F_{1j} $(j = 1, \cdots, N_1)$ is a *similar refinement* of the covering Φ_2 by sets F_{2k} $(k = 1, \cdots, N_2)$ if $N_1 = N_2 = N$ and, for proper numbering, one has $F_{1j} \subseteq F_{2j}$ $(j = 1, \cdots, N)$ and $F_{1j_1} \wedge F_{1j_2} \wedge \cdots \wedge F_{1j_s} \neq 0$ if and only if $F_{2j_1} \wedge F_{2j_2} \wedge \cdots \wedge F_{2j_s} \neq 0$. In this case Φ_1 and Φ_2 are isomorphic complexes.

THEOREM 4. *For each indeterminacy function $\epsilon = \epsilon(x; u)$ there exists a number η, $\eta > 0$, such that, if ϵ_1 and ϵ_2 are indeterminacy functions, with $\epsilon_1 \ll \epsilon_2 \ll \epsilon$ and $\| \epsilon_1 - \epsilon \| < \eta$, $\| \epsilon_2 - \epsilon \| < \eta$ then $\Phi_1 = \Phi(\epsilon_1)$ is a similar refinement of $\Phi_2 = \Phi(\epsilon_2)$.*

Proof. For each λ between 0 and 1 let $N_0(\lambda)$ equal the number of sets (vertices) in the covering $\Phi(\lambda\epsilon)$. As λ increases, $N_0(\lambda)$ is non-increasing.

For, by Theorem 3, each set of $\Phi(\lambda'\epsilon)$ is contained in a set of $\Phi(\lambda''\epsilon)$ for $\lambda' < \lambda''$ and every set of $\Phi(\lambda''\epsilon)$ must appear in this correspondence, since the coverings are minimal. Hence, for λ_0 sufficiently large and < 1, the number $N_0(\lambda)$ is constant for $\lambda_0 \leqq \lambda < 1$, and the refinement correspondence between $\Phi(\lambda'\epsilon)$ and $\Phi(\lambda''\epsilon)$ is one-to-one for $\lambda_0 \leqq \lambda' < \lambda'' < 1$. On the other hand, for $\lambda_0 \leqq \lambda < 1$, the number $N_1(\lambda)$ of edges in $\Phi(\lambda\epsilon)$ must be non-decreasing. For, if two sets in $\Phi(\lambda'\epsilon)$ intersect, the corresponding pair in $\Phi(\lambda''\epsilon)$ also intersect. A similar statement applies to the higher-dimensional faces. Hence, for λ_0 sufficiently large, the number $N_p(\lambda)$ of p-dimensional faces of $\Phi(\lambda\epsilon)$ is constant for all p and for $\lambda_0 \leqq \lambda < 1$. The same reasoning shows that, for this choice of λ_0, the refinement correspondence between $\Phi(\lambda'\epsilon)$ and $\Phi(\lambda''\epsilon)$, for $\lambda_0 \leqq \lambda' < \lambda'' < 1$ satisfies the definition above for a similar refinement.

Now let $\eta = \text{g.l.b.}\ (\epsilon(x;u) - \lambda_0\epsilon(x;u)) = (1-\lambda_0)\ \text{g.l.b.}\ \epsilon(x;u)$, where the g. l. b. is over V'. Then $\eta > 0$ and $\epsilon_2 \ll \epsilon$, $\| \epsilon_2 - \epsilon \| < \eta$ imply $\lambda_0\epsilon \ll \epsilon_2 \ll \lambda'\epsilon$ for λ' sufficiently large and less than 1. This implies that $\Phi(\epsilon_2)$ is a refinement of $\Phi(\lambda'\epsilon)$, while $\Phi(\lambda_0\epsilon)$ is a refinement of $\Phi(\epsilon_2)$. Since $\Phi(\lambda_0\epsilon)$ is a similar refinement of $\Phi(\lambda'\epsilon)$, it follows that $\Phi(\epsilon_2)$ is a similar refinement of $\Phi(\lambda'\epsilon)$. If further $\epsilon_1 \ll \epsilon_2 \ll \epsilon$ and $\| \epsilon_1 - \epsilon \| < \eta$, then $\Phi(\epsilon_1)$ is also a similar refinement of $\Phi(\lambda'\epsilon)$. Since $\Phi(\epsilon_1)$ is also a refinement of $\Phi(\epsilon_2)$, this last refinement must be a similar one. Hence the theorem follows.

Definition. An indeterminacy function $\epsilon(x;u)$ is termed *non-critical* if there exists a number $\zeta > 0$, such that $\| \epsilon_1 - \epsilon \| < \zeta$, $\| \epsilon_2 - \epsilon \| < \zeta$, $\epsilon_1 \ll \epsilon_2$ imply that $\Phi(\epsilon_1)$ is a similar refinement of $\Phi(\epsilon_2)$. If the condition fails, ϵ is called *critical.* It follows from the definition that the non-critical functions form an open set in $\mathcal{E}$.

THEOREM 5. *The critical indeterminacy functions form a nowhere dense set in* $\mathcal{E}$.

Proof. Let ϵ be critical and choose $\eta = \eta(\epsilon)$ as in the preceding theorem. For given $\delta > 0$, one can then choose $\epsilon_3 = \lambda\epsilon$ such that $\epsilon_3 \ll \epsilon$, $\| \epsilon - \epsilon_3 \| < \eta/2$ and $\| \epsilon - \epsilon_3 \| < \delta/2$. Choose $\zeta = \frac{1}{2}\ \text{g.l.b.}\ (\epsilon - \epsilon_3)$ over V'. Then $\zeta > 0$ and $\| \epsilon_j - \epsilon_3 \| < \zeta$ imply $\epsilon_j \ll \epsilon$ and $\| \epsilon_j - \epsilon \| < \eta$ for $j = 1, 2$. Hence, by the preceding theorem, if $\epsilon_1 \ll \epsilon_2$, Φ_1 is a similar refinement of Φ_2. Thus ϵ_3 is non-critical. Since the non-critical functions form an open set, it follows that every neighborhood of ϵ contains an open set of non-critical functions. Thus the critical functions form a nowhere dense set.

5. Approximation by flow on a net.

5.1. Flows in M and in M_0. The purpose of this section is to show that, when ϵ is non-critical and convex, the complex Φ can be found by a finite computation, based on a study of the field $v(x)$ on a finite subset M_0 of the space M. One can first *project* the flow in M onto a flow on M_0 by requiring that $x < y$ hold in M_0 precisely when it holds in M. This clearly determines a partial order in M_0. If this partial order is ω-extensive, one can then apply all the theory of Section 2 above to obtain stationary states, transitory states and (the ω-stability type being necessarily finite because M_0 is finite) the ω-stability complex. Corresponding to a complex Φ_1 in M one thus obtains a complex Φ_1^0 in M_0. It is further clear that, if M_0 is sufficiently dense in M, the ordering in M_0 will be ω-extensive and Φ_1 and Φ_1^0 will be isomorphic. "Sufficiently dense" here means that there must be at least one point of M_0 in each ω-stable state $\pi(x_j)$ and that, whenever an intersection $F_{j_1} \wedge F_{j_2} \wedge \cdots \wedge F_{j_s}$ is non-void, then the intersection contains a point of M_0. Since the sets concerned here are open, a $\delta > 0$ exists such that, if M_0 is δ-dense in M (every point of M within distance δ of a point of M_0), then these conditions hold, and Φ_1 is isomorphic to Φ_1^0.

The process just described does not actually reduce the determination of Φ_1 to a computation on M_0 alone, since the validity of each inequality must be investigated in M. Accordingly, a special process must be devised, involving M_0 alone. The following is a description of the process.

A finite set which is δ_1-dense in M will be called a δ_1-*net*. It is known in differential geometry (cf. H. Seifert and W. Threlfall, *Variationsrechnung im Grossen*, Berlin (1938), p. 49) that there exists a positive number d such that each two points of M within distance d can be joined by a unique minimizing geodesic. Let δ_2 satisfy: $0 < \delta_2 < d$. If M_0 is a δ_1-net and each two points of M_0 within distance δ_2 have been joined by the corresponding geodesic, then M_0 becomes a $\delta_1\delta_2$-*network*. The points of M_0 will be numbered as z_μ $(\mu = 1, \cdots, m)$ and the geodesic arcs will be denoted by $C_{\mu\nu}$ for appropriate values of μ and ν. The given vector v at z_μ will be denoted by v_μ and the unit tangent vector to $C_{\mu\nu}$ at z_μ in the direction of z_ν will be denoted by $u_{\mu\nu}$.

Let $\epsilon^0(z_\mu; u)$ now be an indeterminacy function in M_0, i.e., a function satisfying (2) and continuous in u for $u \neq 0$ and z_μ in M_0. Let $C_{\mu\nu}$ be an arc of the $\delta_1\delta_2$-network. If, for some $k > 0$,

$$\| ku_{\mu\nu} - v_\mu \| < \epsilon^0(z_\mu; ku_{\mu\nu} - v_\mu), \tag{6}$$

then z_ν will be termed a *successor of* z_μ. If $z_{\mu_1}, z_{\mu_2}, \cdots, z_{\mu_p}$ are points of M_0 such that z_{μ_k} is a successor of $z_{\mu_{k-1}}$ for $k = 2, \cdots, p$, then the inequality: $z_{\mu_1} \prec z_{\mu_p}$ will be said to hold and the succession of arcs $C_{\mu_1\mu_2}, C_{\mu_2\mu_3}, \cdots, C_{\mu_{p-1}\mu_p}$ together form an *allowed* ϵ^0-*trajectory* on the $\delta_1\delta_2$-network. This definition determines a partial order in M_0. If this partial order is ω-extensive, the theory of Section **2** can be applied, the order has finite stability type, and a complex Φ^0 is determined. The main goal of this section is to establish the following theorem.

THEOREM 6. *Let* ϵ_1 *be a non-critical convex indeterminacy function, of class* C' *in* u *for each* x, $u \neq 0$. *Let* M_0 *be a* $\delta_1\delta_2$-*network based on the points* z_μ $(\mu = 1, \cdots, m)$. *Then, for* δ_1, δ_2, *and* δ_1/δ_2 *sufficiently small and* $\epsilon^0 = \epsilon_1(z_\mu; u)$, *the ordering* $z_\mu \prec z_\nu\ [\epsilon^0]$ *is* ω-*extensive and the complex* Φ^0 *is isomorphic to the complex* Φ_1.

5. 2. Two fundamental lemmas. In order to prove the theorem just stated, one first chooses constants λ_2 and λ_3 so that $\epsilon_2 = \lambda_2\epsilon_1 \ll \epsilon_1 \ll \epsilon_3 = \lambda_3\epsilon_1$ and $\| \epsilon_2 - \epsilon_1 \| < \zeta(\epsilon_1)$, $\| \epsilon_3 - \epsilon_1 \| < \zeta(\epsilon_1)$. It follows that the complexes $\Phi_j = \Phi(\epsilon_j)$ $(j = 1, 2, 3)$ are isomorphic, the covering Φ_2 being a similar refinement of Φ_1, the covering Φ_1 being a similar refinement of Φ_3. Let M_0 be a $\delta_1\delta_2$-network, with δ_1 so small that the projections $\Phi_2{}^0$ and $\Phi_3{}^0$ are defined and are isomorphic to Φ_2 and Φ_3 respectively; hence $\Phi_2{}^0$ and $\Phi_3{}^0$ are isomorphic, and $\Phi_2{}^0$ is a similar refinement of $\Phi_3{}^0$.

LEMMA 1. *For* δ_1, δ_2 *and* δ_1/δ_2 *sufficiently small,* $z_\mu < z_\nu\ [\epsilon_2]$ *implies* $z_\mu \prec z_\nu\ [\epsilon^0]$.

LEMMA 2. *For* δ_1 *and* δ_2 *sufficiently small,* $z_\mu \prec z_\nu\ [\epsilon^0]$ *implies* $z_\mu < z_\nu\ [\epsilon_3]$.

The proofs of these lemmas will be given in the following sections. Here it will be shown that they imply Theorem 6.

By the above construction, δ_1 is chosen so small that the ordering $z_\mu < z_\nu\ [\epsilon_2]$ is ω-extensive on M_0. Lemma 1 then implies that the ordering $z_\mu \prec z_\nu\ [\epsilon^0]$ is ω-extensive on M_0, so that a complex Φ^0 is obtained. By virtue of Lemma 1, the covering $\Phi_2{}^0$ is a refinement of the covering Φ^0. By virtue of Lemma 2, the covering Φ^0 is a refinement of the covering $\Phi_3{}^0$. Since $\Phi_2{}^0$ is a similar refinement of $\Phi_3{}^0$, it follows that $\Phi_2{}^0$ is a similar refinement of Φ^0 and that Φ^0 is a similar refinement of $\Phi_3{}^0$. Hence $\Phi_2{}^0$, Φ^0, $\Phi_3{}^0$ are all isomorphic. Since $\Phi_2{}^0$ is isomorphic to Φ_2, and Φ_2 to Φ_1, the theorem is thus established.

5.3. Auxiliary lemmas. Before proceeding to the proofs of Lemmas 1 and 2, it will be useful to carry out several preliminary steps which will simplify the uniformity arguments to follow.

At each phase x_0 of M a coordinate system can be chosen for a neighborhood of x_0 and an $r > 0$ can then be chosen so that the Euclidean sphere S: $\Sigma(x^i - x_0^i)^2 \leqq r^2$, in these coordinates, lies within the coordinate neighborhood. Further, a number $R > 0$ can then be chosen so that the neighborhood $\rho(x_0, x) < R$ lies in S. The collection of all neighborhoods $W: \rho(x_0, x) < \frac{1}{2}R$ covers M. Hence a finite number of these: $W_q: \rho(y_q, x) < \frac{1}{2}R_q$ $(q = 1, \cdots, Q)$ covers M. With each W_q there is associated a coordinate system and a Euclidean sphere S_q in these coordinates such that the neighborhood $\rho(y_q, x) < R_q$ lies in S_q. In the following proof only these coordinate systems will be used.

Let $R_0 = \min(R_1, \cdots, R_Q)$. If $\rho(z_\mu, z_\nu) < d$ and $\rho(z_\mu, z_\nu) < R_0/4$, then $\rho(z_\mu, y_q) < \frac{1}{2}R_q$ and $\rho(z_\nu, y_q) < 3R_q/4$ for some q. Hence both the geodesic $C_{\mu\nu}$ and the Euclidean straight line, in terms of the chosen coordinates at y_q, lie within S_q.

Throughout the following it will often be convenient to use the E-function (3.2 above) corresponding to a given indeterminacy function $\epsilon(x; u)$. In each case the matching of subscripts or superscripts will indicate the corresponding pairs of functions.

LEMMA 3. *Let ϵ_1 and ϵ_2 be indeterminacy functions in M such that $\epsilon_1 \ll \epsilon_2$. Then there exists a constant h such that, if x_1 and x_2 are in S_q and $(x_1; u_1)$, $(x_2; u_2)$ are such that*

$$E_1(x_1; u_1 - v(x_1)) < 1, \quad | x_1^i - x_2^i | < h, \quad | u_1^i - u_2^i | < h$$

in the corresponding coordinates, then $E_2(x_2; u_2 - v(x_2)) < 1$.

Proof. Let I_j $(j = 1, 2)$ denote the subset of V consisting of all elements $(x; w)$ such that $E_j(x; w - v(x)) < 1$. Then, since the E_j and v are continuous, I_j is open. Further, because of the homogeneity of the E_j, the closure $\bar{I}_1$ of I_1 is compact and lies in I_2. Hence the distance σ_0 between I_1 and the boundary of I_2 is positive and, if $(x_1; u_1)$ is in I_1 and $\sigma((x_1; u_1), (x_2; u_2)) < \sigma_0$, then $(x_2; u_2)$ is in I_2. Because of the compactness of $\bar{I}_1$ and the equivalence of the topology in V with the Euclidean topology defined by the coordinate system in S_q, a number h_q can be found such that, if $(x_1; u_1)$ is in I_1 and $| x_1^i - x_2^i | < h_q$, $| u_1^i - u_2^i | < h_q$ then $\sigma((x_1; u_1), (x_2; u_2)) < \sigma_0$. If h is now chosen as $\min(h_1, \cdots, h_Q)$, then the conclusion follows.

The geodesics in M are defined by equations of the form:

(7) $$d^2x^i/ds^2 = f^i(x^1, \cdots, x^n, dx^1/ds, \cdots, dx^n/ds),$$

where the f^i are continuous. The variables dx^i/ds are the components of a unit vector and hence are bounded (in absolute value), in the local coordinates, in S_q by a constant H_q. Thus $H = \max(H_1, \cdots, H_Q)$ is a bound for these components on M. In each set S_q, the point $(x^1, \cdots, x^n, dx^1/ds, \cdots, dx^n/ds)$ ranges over a compact subset of V. Hence the components f^i are uniformly bounded in S_q by a constant F_q and, if $F = \max(F_1, \cdots, F_Q)$, then F serves as a uniform bound for these components on M. Similarly, the coefficients a_{ij} of the fundamental form ds^2 are bounded by a constant A on M.

LEMMA 4. *Let x_1x_2 be a geodesic in M of length s_1 lying completely in a set S_q. Then one has*

(8) $$| x_2^i - x_1^i | < Fs_1^2 + Hs_1,$$
$$| u_2^i - u_1^i | < Fs_1, \; | u_1^i - (x_2^i - x_1^i)/s_1 | < Fs_1^2,$$

where $u_j = dx/ds$ at x_j.

Proof. The differential equations (7) above can be integrated once along x_1x_2 to give

(9) $$dx^i/ds = \int_0^s f^i ds + u_1{}^i,$$

and again to give

(10) $$x^i = \int_0^s \int_0^s f^i ds ds + u_1{}^i s + x_1{}^i.$$

The inequalities (8) follow at once from (9) and (10) and from the fact that $| f^i | < F$, $| u_j{}^i | < H$.

LEMMA 5. *Let D denote the n-dimensional space of coordinates $(x^1, \cdots, x^n)$ with metric $ds^2 = a_{ij}dx^i dx^j$, the a_{ij} being constants. Let $v(x)$ be a vector field defining a dynamical system in D, whereby $v(x)$ is independent of x, so that $v(x) = v(0)$. Let $\epsilon(x; u)$ be a convex indeterminacy function in D, of class C' for $u \neq 0$, which is independent of x, so that $\epsilon(x; u) = \epsilon(u)$. Let y and z be in D and let $y < z$ [ϵ]. Let yz be an allowed ϵ-trajectory from y to z, and let τ be the time interval on this trajectory. Then the uniform trajectory:*

(11) $$x^i = y^i + (z^i - y^i)t/\tau \qquad (i = 1, \cdots, n)$$

from y to z is also an allowed ϵ-trajectory from y to z.

Proof. The equations

$$x^i = y^i + t(v^i + u^i \varepsilon(u)), t \geqq 0, \tag{12}$$

where u is a variable unit vector, define a semi-cone in $x^1 \cdots x^n t$-space. By the "inside" of this cone will be meant the domain

$$x^i = y^i + t(v^i + w^i \epsilon(w)), t > 0, \tag{13}$$

where w is a variable vector with $\| w \| < 1$. The trajectory yz corresponds to a curve

$$x^i = \phi^i(t), \qquad t = t \qquad (0 \leqq t \leqq \tau) \tag{14}$$

in $x^1 \cdots x^n t$-space, passing through the vertex $(y^1, \cdots, y^n, 0)$ of the cone.

Now the function $E(x; u) = E(u)$ corresponding to $\epsilon(u)$ is positive, homogeneous of degree one, and the sets $E \leqq$ const. are convex. From these two properties one concludes that, for any points x and z $(x \neq 0)$ one has

$$z^i \partial E / \partial x^i \leqq E(z) \,. \tag{15}$$

For if z lies in the tangent plane to the surface $E =$ const. at x, then $(x^i - z^i)\partial E/dx^i = 0$ or, by the Euler theorem, $E(x) = z^i \partial E/\partial x^i$. Since z cannot lie inside the surface $E =$ const., one has $E(x) \leqq E(z)$, so that (15) holds. The homogeneity of $E(z)$ then implies (15) for arbitrary z.

Now let $g(x, t)$ be defined by the equation

$$g(x, t) = E(x - y - vt). \tag{16}$$

Then on the cone (12) one has

$$g(x, t) = \| x - y - vt \| / \{\epsilon(x - y - vt)\} = t\epsilon(u)/\{\epsilon(tu\epsilon(u))\} = t, \tag{17}$$

and, by a similar reasoning, $g(x, t) > t$ for $t > 0$ and (x, t) outside the cone (12). On the path (14) one has $E(dx/dt - v(x)) < 1$. Hence, by (15), at each point on this path $x = y + vt$, or

$$dg/dt = (dx^i/dt - v^i)\partial E/\partial x^i \leqq E(dx/dt - v) < 1. \tag{18}$$

It follows that the path (14) enters the interior of the cone at $t = 0$ and remains inside. Hence the end-point z is inside the cone, and the straight

line path (11) is inside the cone. Hence this path can be written in the form (13) with constant w, $\| w \| < 1$. Hence on (11)

$$(19) \qquad \| dx/dt - v \| = \| w \| \epsilon(w) < \epsilon(w) = \epsilon(dx/dt - v).$$

Thus (11) is an allowed ϵ-trajectory, and Lemma 5 is established.

Remark. The assumption that $\epsilon(u)$ is of class C' is clearly needed for the proof just given, and this is the basis for the corresponding assumption in Theorem 6. It is however possible to prove, without too much trouble, that a convex indeterminacy function $\epsilon(u)$ can be arbitrarily closely approximated by a convex function of class C'. By means of this approximation, Lemma 5 can be applied to prove Theorem 6 without the assumption that $\epsilon_1(x; u)$ be of class C'. The proof is given in the present form for the sake of simplicity.

LEMMA 6. *Let $\epsilon(x; u)$ be a given indeterminacy function in M. Then there exist constants B_1, B_2, C such that*

$$(20) \qquad | dx^i/dt | < B_1, \qquad | dx^i/dt - v^i | < B_2, \qquad s < C\tau,$$

in the coordinates of each set S_q, where $x^i = x^i(t)$, $t_1 \leqq t \leqq t_1 + \tau$, is an ϵ-solution from x_1 to x_2 of length s in S_q.

Proof. The function $\epsilon(x; u)$ is uniformly bounded by a constant ϵ^*. Hence $\| dx/dt - v(x) \| < \epsilon^*$. The set of all $(x; w)$ such that x is in S_q and $\| w \|^2 = a_{ij} w^i w^j \leqq \epsilon^{*2}$ is a compact subset of V. Hence its projection on the w^i-axis is bounded. Hence a constant B_2 can be chosen, independent of q, so that $\| w \|^2 \leqq \epsilon^{*2}$ implies $| w^i | < B_2$. Thus on the ϵ-trajectory $| dx^i/dt - v^i | < B_2$. The vector v has also a maximum norm K_1 on M. Hence there is a constant K_2 (independent of q), such that in each S_q $| v^i | < K_2$. Hence $| dx^i/dt | < K_2 + B_2 = B_1$. The velocity on an ϵ-solution is bounded by $K_1 + \epsilon^* = C$. Hence s/τ, which is simply the average velocity, is bounded by C.

5.4. Proof of Lemma 1. *Remark.* Throughout the following proofs the choices of δ_1 and δ_2 will be continually restricted. In each case the restriction will be a uniform one, i. e., dependent only on the given vector field $v^i(x)$ and indeterminacy function $\epsilon_1(x; u)$ and not on the particular points or neighborhoods considered.

Let $\epsilon_4 = \lambda_4\epsilon_1$ and $\epsilon_5 = \lambda_5\epsilon_1$ be chosen so that $\epsilon_2 \ll \epsilon_4 \ll \epsilon_5 \ll \epsilon_1$. Choose δ_2 less than $R_0/4$.

Suppose now that $z_\mu < z_\nu\ [\epsilon_2]$, so that there is an allowed ϵ_2-solution in M from z_μ to z_ν.

Case A. *The ϵ_2-solution $z_\mu z_\nu$ from z_μ to z_ν lies within the δ_2-neighborhood of z_μ.* One can then choose local coordinates in a sphere S_q as above, with $\rho(z_\mu, y_q) < R_q/2$.

It follows from Lemma 3 that, for δ_2 sufficiently small, one has at each phase x of the trajectory $z_\mu z_\nu$ $E_4(z_\mu; dx/dt - v(x)) < 1$, so that $z_\mu z_\nu$ is an allowed solution relative to the constant (independent of x) indeterminacy function $\epsilon' = \epsilon_4(z_\mu; u)$ in S_q. Thus one has at each point of $z_\mu z_\nu$

$$\| dx/dt - v(x) \| < \epsilon'(dx/dt - v(x)). \tag{21}$$

Since the a_{ij} are continuous, one can further choose δ_2 so small that at each point of $z_\mu z_\nu$ one has

$$\| dx/dt - v(x) \|_\mu < \epsilon(dx/dt - v(x)), \tag{22}$$

where $\| w \|_\mu = (a_{ij}(z_\mu) w^i w^j)^{\frac{1}{2}}$ and $\epsilon(w) = \epsilon_5(z_\mu; w)$. For let $\xi_1 = \text{g. l. b.}$ $(\epsilon_5(x; u) - \epsilon_4(x; u))$ on V', so that $\xi_1 > 0$. Given $\xi_2 > 0$, then δ^* can be chosen so that $| a_{ij}(x) - a_{ij}(z_\mu) | < \xi_2$ for $\rho(x, z_\mu) < \delta^*$, whereby δ^* can be chosen independent of x and the particular S_q. Hence, for $\delta_2 < \delta^*$,

$$\begin{aligned} &| \, \| dx/dt - v \|^2 - \| dx/dt - v \|_\mu^2 \, | \\ &\qquad = | (a_{ij}(x) - a_{ij}(z_\mu)) (dx^i/dt - v^i)(dx^j/dt - v^j) | \\ &\qquad\qquad < \xi_2 \sum_{i,j} | dx^i/dt - v^i | \, | dx^j/dt - v^j | < \xi_2 n^2 B_2{}^2, \end{aligned}$$

where the last inequality follows from Lemma 6, since $z_\mu z_\nu$ is an ϵ_2-solution. Since in general $| \, |a| - |b| \, | \leqq (| a^2 - b^2 |)^{\frac{1}{2}}$, one concludes that

$$| \, \| dx/dt - v \| - \| dx/dt - v \|_\mu \, | < n B_2 \xi_2^{\frac{1}{2}},$$

hence

$$\| dx/dt - v \|_\mu < \epsilon'(dx/dt - v) + n B_2 \xi_2^{\frac{1}{2}},$$

by (21). If now ξ_2 is chosen so that one has $n B_2 \xi_2^{\frac{1}{2}} < \xi_1$ and δ^* is chosen accordingly, then (22) follows.

Thus $z_\mu z_\nu$ is an ϵ-solution in $x^1 \cdots x^n$-space, with constant metric, where $\epsilon(w) = \lambda_5 \epsilon_1(z_\mu; w)$ is of class C' for $w \neq 0$ and convex. Hence Lemma 5

can be applied and one concludes that the uniform trajectory from z_μ to z_ν in S_q is also an ϵ-solution. Thus in particular one has at z_μ (in local coordinates)

$$\|(z_\mu - z_\nu)/\tau - v_\mu\| < \epsilon_5(z_\mu; (z_\mu - z_\nu)/\tau - v_\mu), \tag{23}$$

where τ is the time on $z_\mu z_\nu$.

Now the geodesic $C_{\mu\nu}$ has length $s < \delta_2$ and one has thus, by Lemma 4,

$$|(z_\mu{}^i - z_\nu{}^i)/s - u_{\mu\nu}{}^i| < F\delta_2{}^2. \tag{24}$$

Hence

$$|(z_\mu{}^i - z_\nu{}^i)/\tau - u_{\mu\nu}{}^i(s/\tau)| < F\delta_2{}^2C. \tag{25}$$

For s is at most equal to the length of $z_\mu z_\nu$, so that s/τ is at most equal to C, in accordance with Lemma 6. It now follows from (23) and (25) and Lemma 3 that, for δ_2 sufficiently small one has

$$E_1(z_\mu; u_{\mu\nu}(s/\tau) - v_\mu) < 1. \tag{26}$$

Thus $z_\mu \prec z_\nu$ $[\epsilon^0]$. Thus Lemma 1 is established in Case A.

Case B. *z_ν does not lie within the δ_2-neighborhood of z_μ.* Then proceed along $z_\mu z_\nu$ from $z_\mu = p_0$ until a point p_1 is reached at distance $\delta_2/3$ from z_μ. Proceed from p_1 until a point p_2 at distance $\delta_2/3$ from p_1 is reached, and so on. One will finally reach a point p_k such that there is no point beyond p_k at distance greater than $\delta_2/3$ from p_k. If $\rho(p_k, z_\nu) < \delta_2/6$, then replace the previous choice of p_k by z_ν, so that one now has $\rho(p_{k-1}, p_k) > \delta_2/6$ and further the trajectory $p_{k-1}p_k$ lies within the $2\,\delta_2/3$ neighborhood of p_{k-1}. If $\rho(p_k, z_\nu) \geqq \delta_2/6$, then set $z_\nu = p_{k+1}$. Thus in either case one has a sequence of points on $z_\mu z_\nu : z_\mu = p_0, p_1, \cdots, p_h = z_\nu$ such that on each sub-arc $p_j p_{j+1}$ one has the condition: $\rho(p_j, p_{j+1}) \geqq \delta_2/6$, while the arc $p_j p_{j+1}$ lies within the $2\,\delta_2/3$ neighborhood of p_j.

Now each arc $p_j p_{j+1}$ lies within one of the sets S_q. Proceeding as in Case A, one can then show that, for δ_2 small enough, the uniform trajectory from p_j to p_{j+1}, in the local coordinates, is an ϵ_4-solution.

Now there is by assumption a point z_{μ_j} of M_0 within distance δ_1 of each p_j. Let δ_1 be chosen less than $\delta_2/6$ and less than $R_0/4$. Then the set S_q can be chosen to contain not only the trajectory $p_j p_{j+1}$, but also the points z_{μ_j}, $z_{\mu_{j+1}}$ and the two corresponding line segments in the local coordinates. Let τ be the time used on the trajectory $p_j p_{j+1}$. Then one has

$$E_4(x; [p_{j+1} - p_j]/\tau - v(x)) < 1 \tag{27}$$

at each point x of the segment p_jp_{j+1}. One has further, by Lemma 4,

$$(28)\qquad |z^i_{\mu_j} - p^i_j| < F\delta_1^2 + H\delta_1 < L\delta_1, \qquad |z^i_{\mu_{j+1}} - p^i_{j+1}| < L\delta_1$$

for a constant L. Furthermore,

$$(29))\qquad \begin{aligned} &|(z^i_{\mu_{j+1}} - z^i_{\mu_j})/\tau - (p_{j+1} - p^i_j)/\tau| \\ &\leqq \tau^{-1}(|z^i_{\mu_{j+1}} - p^i_{j+1}| + |z^i_{\mu_j} - p^i_j|) \leqq 2L\delta_1/\tau \leqq 2L\delta_1(6C/\delta_2). \end{aligned}$$

For $\rho(p_j, p_{j+1}) \geqq \delta_2/6$, so that, by Lemma 6, the time τ is greater than $\delta_2/6C$. Now let $x = \phi_1(t)$, $t_1 \leqq t \leqq t_1 + \tau$ represent the uniform trajectory from p_j to p_{j+1} and let $x = \phi_2(t)$, $t_1 \leqq t \leqq t_1 + \tau$ represent the uniform trajectory, with the same interval τ, from z_{μ_j} to $z_{\mu_{j+1}}$. Then (28) and (29) imply that there is a constant h such that

$$\sigma((\phi_1(t); d\phi_1/dt), (\phi_2(t); d\phi_2/dt)) < h,$$

and h can be made as small as desired by choosing δ_1 and δ_1/δ_2 sufficiently small. One concludes from Lemma 3 and (27) that, for δ_1 and δ_1/δ_2 sufficiently small, the uniform trajectory $x = \phi_2(t)$ is an ϵ_5-solution. It now follows as in Case A that $z_{\mu_j} \prec z_{\mu_{j+1}} [\epsilon^0]$. Thus, finally, $z_\mu \prec z_\nu [\epsilon^0]$, for z_{μ_1} can be chosen as z_μ, and z_{μ_k} can be chosen as z_ν, and the transitive law can then be applied. Thus Lemma 1 is completely established.

5.5. Proof of Lemma 2. Suppose first that z_ν is a successor of z_μ. One is then given that, for some $k > 0$,

$$(30)\qquad E_1(z_\mu; ku_{\mu\nu} - v_\mu) < 1.$$

Choose local coordinates in a set S_q containing $C_{\mu\nu}$ as above. Then, by Lemma 4, one has $|u^i - u^i_{\mu\nu}| < F\delta_2$, where u is the tangent vector dx/ds at an arbitrary point x of $C_{\mu\nu}$. Now the values k for which (30) can hold are uniformly bounded on M by a constant k_0. Hence $|ku^i - ku^i_{\mu\nu}| < k_0F\delta_2$. Since the vector field $v(x)$ is continuous, one can further, by Lemma 4, choose δ_2 so small that $|v^i - v^i_\mu| < \eta_0$, for given $\eta_0 > 0$, where $v = v(x)$ at an arbitrary point x on $C_{\mu\nu}$. Hence $|(ku^i - v^i) - (ku^i_{\mu\nu} - v^i_\mu)| < k_0F\delta_2 + \eta_0$. Hence, by Lemma 3, if δ_2 is sufficiently small, one has $E_3(x; ku(x) - v(x)) < 1$ at each point x of $C_{\mu\nu}$. It follows that the path $x = x(t)$, from z_μ to z_ν along $C_{\mu\nu}$, such that $dx/dt = ku(x)$ is an ϵ_3-solution, i. e., that $z_\mu < z_\nu [\epsilon_3]$, as was to be shown. If z_ν is not a successor of z_μ, the same conclusion follows by consideration of each link in the chain from z_μ to z_ν. Thus Lemma 2 is established.

6. Case of non-compact M. Remarks on applications.

6.1. Case of non-compact M. It is clear that the compactness of the phase space M plays an important role in the above discussion. If M is not compact, the number of stable stationary states may be infinite, and the approximation by a flow on a finite set is in general not possible.

There are, however, circumstances, of considerable physical interest, under which the above results can be extended to the case of a non-compact manifold M.

First of all, it may happen that there is a compact subset M_1 of M such that each allowed ϵ-trajectory $x = x(t)$ enters and remains in M for t sufficiently large. In this case the discussion of ω-stability can be restricted to M_1 and the previous results can be extended without difficulty.

Secondly, it may be possible to imbed M in a compact space M' in such a manner that the vector field v can be extended continuously to M'. The relationships between the states in M' and those in M will in general not be simple, but the analysis in M' will give some information about the flow in M.

Finally, it may happen that, outside of a compact subset M_1 of M, the given indeterminacy function $\epsilon(x; u)$ has the property that $\epsilon(x; -v) > \| v \|$, so that the null vector is in the indeterminacy neighborhood at each x of $M - M_1$. This implies that in $M - M_1$ every direction is allowed, so that each component of $M - M_1$ forms a stationary state. Thus the analysis in $M - M_1$ is simple, and the problem is effectively restricted to M_1.

This last case can be interpreted physically as follows. For a particular physical system the accuracy of one's knowledge will usually diminish as one recedes from a given bounded part of the phase space. Thus physical knowledge concerning the cases of extremely large distances or extremely large velocities is very limited. This is reflected in the above condition on the indeterminacy function $\epsilon(x; u)$, for the condition simply states that one is wholly uncertain as to what course the system will take outside of M_1.

6.2. Applications. The approximation theorem of Section **5** above was developed with particular applications in "non-linear mechanics" in mind, and a number of such problems have been studied from this point of view. It is intended to publish the results in a separate paper. Here some remarks will be made on the methods used and one illustration, a case of the Van der Pol equation, will be given.

In the problems considered, the phase space M was either Euclidean n-space or a product of such a space by a circle. Thus M was non-compact. In all cases either the first or third point of view of Section **6.1** was

applicable, so that one could effectively restrict attention to a compact subset M_1.

Suppose then that M is Euclidean n-space, and that differential equations (1) are given in M. The indeterminacy function $\epsilon_1(x;u)$ may then be given in advance or may be determined at a later stage. The latter procedure is usually more convenient, as will be made clear. One now chooses as M_0 the set of all points $(m_1h,\cdots,m_nh)$, where the number $h>0$ is fixed and the numbers m_i are integers. At each such point x one defines the "first shell" of neighboring points as those points of M_0, other than x, whose coordinates differ from those of x by at most h. Thus, for $n=2$, there are 8 first shell neighbors: $(m_1\pm h, m_2h)$, $(m_1, m_2\pm h)$, $(m_1\pm h, m_2\pm h)$. In general, the p-th shell of neighboring points of x is defined as those points of M_0, other than x or those of the first $p-1$ shells, whose coordinates differ from those of x by at most ph. Thus, for $n=2$, there are $8p$ p-th shell neighbors.

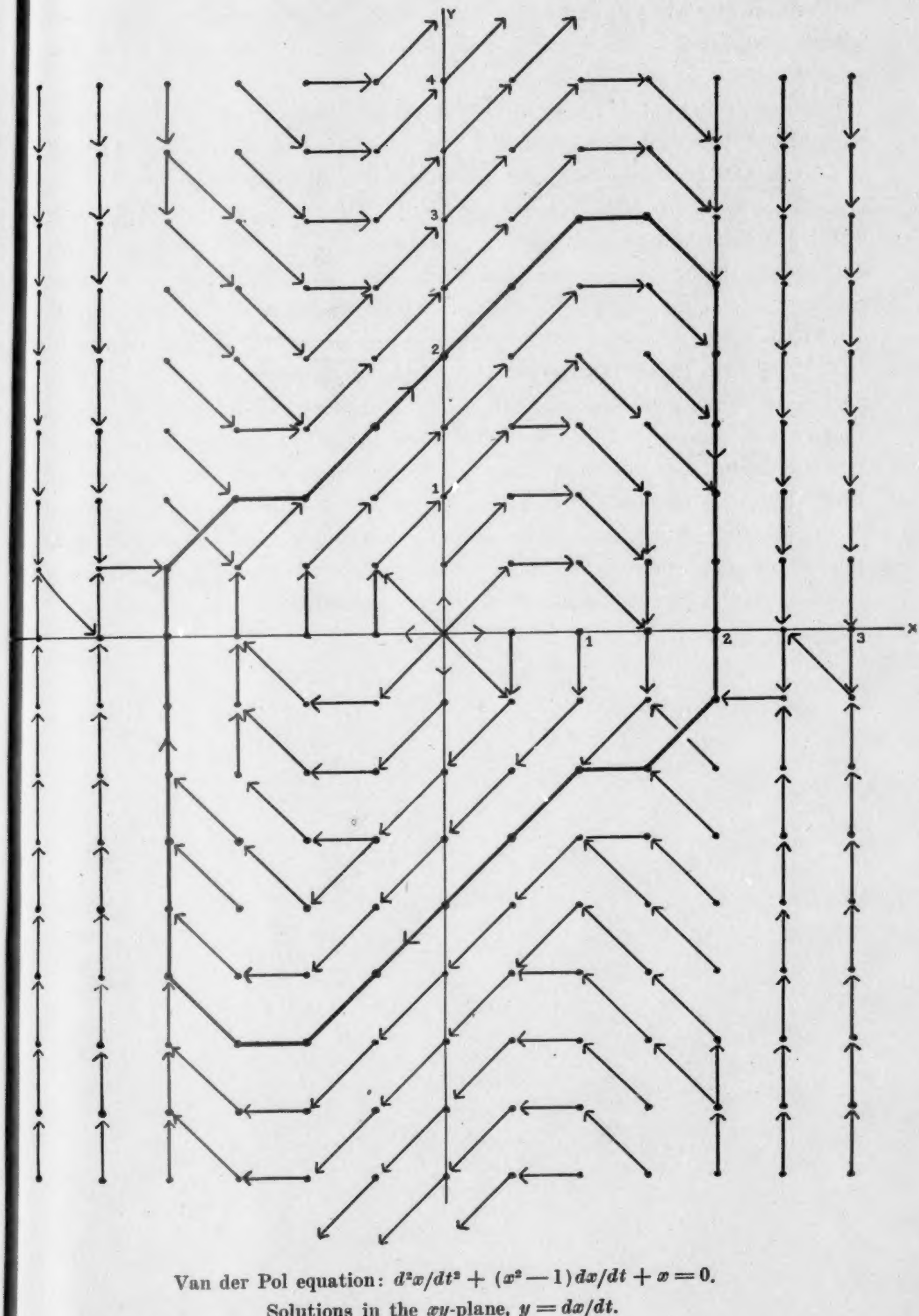
One now chooses a particular value of p, and for each x of M_0 chooses successors only among the points of the first p shells. Fixing h and p is clearly equivalent to fixing the values of δ_1 and δ_2 as in Section **5**. The geodesics are now simply line segments.

If $\epsilon_1(x;u)$ is known, one now chooses as successors of x all points y of the first p shells such that the vector u_{xy}, represented by the line segment xy, satisfies $\| ku_{xy}-v(x)\| < \epsilon_1(x; ku_{xy}-v(x))$ for some $k>0$. This condition restricts the choice of u_{xy} to a certain cone of directions about that of v.

Usually, however, $\epsilon_1(x;u)$ is not given in advance. One can then effectively choose an ϵ_1 by choosing as successors of x all points y of the first p shells such that the vector u_{xy} makes an angle with v less than a prescribed amount θ. Thus, for $n=2$ and $p=1$, one could simply choose all y such that the direction from x to y differs from that of $v(x)$ by less than $23°$, so that each x has at least one successor.

Many variations on these procedures are possible. In particular, one can always adjust the $\epsilon_1(x;u)$ to reflect differences in degree of indeterminacy along the directions of the coordinate axes.

Whatever procedure is used, one will obtain a partial order or flow on M_0, and the stable states and complex Φ^0 will be determined as above. The question whether this complex is isomorphic to the complex Φ_1 for the 'corresponding degree of indeterminacy' $\epsilon_1(x;u)$ in M cannot be answered in general. The approximation theorem asserts that this must be the case if ϵ_1 is convex and non-critical and if δ_1, δ_2 and δ_1/δ_2 are sufficiently small (p sufficiently large, h and ph sufficiently small). However, no effective procedure is given for determining when these conditions are met, in particular



Van der Pol equation: $d^2x/dt^2 + (x^2 - 1)\,dx/dt + x = 0$.
Solutions in the xy-plane, $y = dx/dt$.

for determining when δ_1 and δ_2 are properly chosen. In most particular cases studied thus far, it has been easy to determine when the analysis is fine enough.

As an example, the accompanying figure illustrates the result of applying the method described to the Van der Pol equation: $d^2x/dt^2 + (x^2 - 1)dx/dt + x = 0$. One replaces this by the system $dx/dt = y$, $dy/dt = (1 - x^2)y - x$, and the phase space M is then the xy-plane. The set M_0 is the set of points $(\frac{1}{2}m_1, \frac{1}{2}m_2)$ for integral m_1, m_2. At each point (x, y) the successor (x', y') is chosen in the first shell as the point whose direction differs from the assigned one by less than $23°$. Thus, in general at least one successor is obtained. At $(0, 0)$ the vector v reduces to 0, so that the direction is wholly indeterminate.

The figure shows the results of analysis of the rectangle $M_1 : |x| \leqq 3$, $|y| \leqq 4$. Outside of this rectangle one can consider the directions as wholly indeterminate, in accordance with the third method of Section **6. 1**. If this is done, then one concludes at once that there is one ω-stable state, represented by the heavy simple closed curve. This result agrees with the known properties of the solutions, and the ω-stable state is a reasonably good approximation to the known stable periodic solution. Thus the introduction of indeterminacy in the vector field has not distorted the problem significantly.

QUELQUES PROPRIÉTÉS EXTRÉMALES DU CERCLE ET DE LA SPHÈRE.*

Par Robert Sips.

1. Introduction. Le cercle et la sphère possèdent une série de propriétés isopérimétriques sans relations apparentes les unes avec les autres, mais qui présentent cependant ce caractère commun de ne pouvoir être démontrées par les procédés habituels du calcul des variations.

La plus anciennement connue de ces propriétés se rapporte aux membranes et a été énoncée sans démonstration par Lord Rayleigh [1]. D'après celui-ci, parmi toutes les membranes de même aire et soumises à une même tension uniforme, la membrane circulaire a la fréquence fondamentale la plus basse. Ceci a été démontré pour la première fois par G. Faber [2]. Une autre démonstration a été donnée depuis par E. Krahn [3].

R. Courant [4] a démontré que, parmi toutes les membranes de même longueur périphérique, la membrane circulaire avait la fréquence fondamentale la plus basse.

Une autre propriété extrémale de la sphère a été trouvée par Liapounoff, lequel a démontré que, pour une masse fluide de volume donné, l'énergie potentielle maximum correspondait à la forme sphérique. D'autres démonstrations de cette propriété ont été données par H. Poincaré [5] et par T. Carleman [6].

T. Carleman [7] a démontré que, parmi tous les condensateurs constitués par deux cylindres parallèles indéfinis dont les sections droites ont des aires données, celui constitué par deux cylindres circulaires concentriques a la capacité minimum par unité de longeur.

Ce théorème a été étendu par G. Szegö [8] au cas du condensateur formé par deux surfaces fermées, dont l'une entoure complètement l'autre. La capacité minimum correspond au cas de deux sphères concentriques.

G. Pólya [9], enfin, a prouvé que le module de torsion d'un prisme dont la section droite a une aire donnée est le plus grand possible pour le prisme de section circulaire. Ses résultats ont été généralisés par G. Pólya et A. Weinstein [10].

Dans ce qui suit, nous nous proposons, en généralisant la méthode de

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G. Szegö, de montrer que toutes les propriétés énumérées plus haut, ainsi que quelques d'autres que nous croyons nouvelles, peuvent être considérées comme des cas particuliers d'une propriété isopérimétrique générale des systèmes de cercles et de sphères concentriques.

2. Inégalité fondamentale. Considérons, dans le plan (x, y) un domaine (D) limité intérieurement par un nombre limité n de courbes fermées extérieures les unes aux autres: $C_{i,1}, C_{i,2}, \cdots, C_{i,n}$, et extérieurement par une courbe fermée C_e qui entoure complètement, sans avoir de points communs avec elles, les courbes C_i. En particulier, C_e peut être située entièrement à l'infini.

Les courbes C_i et C_e seront constituées chacune par un nombre fini d'arcs réguliers et ne possèderont pas de points multiples. L'une ou plusieurs des courbes C_i peuvent d'ailleurs se réduire à un point.

Soit maintenant $z(x, y)$ une solution de l'équation différentielle

$$z_{xx} + z_{yy} = \Delta^2 z = f(z) \tag{1}$$

finie et continue ainsi que ses dérivées partielles des deux premiers ordres dans (D), qui prend la valeur constante z_i sur les courbes C_i et la valeur constante $z_e > z_i$ sur la courbe C_e. Lorsque $z_i \leqq z \leqq z_e$, nous admettrons que $f(z)$ est une fonction analytique positive, bornée et continue de z.

La fonction $z(x, y)$ sera alors une fonction analytique dans tout le domaine (D), sauf peut-être sur les frontières C_i et C_e.

Si nous considérons z comme une troisième coordonnée perpendiculaire au plan des (x, y), l'équation $z = z(x, y)$ représentera une surface dont les courbes de niveau seront les courbes $z =$ const. Nous n'aurons évidemment à considérer que la portion de cette surface dont les points ont leur projection contenue dans (D). Les dérivées partielles z_x et z_y étant partout finies, les courbes de niveau ne pourront avoir de points communs. De plus, $f(z)$ étant par hypothèse toujours positif, la surface ne pourra posséder de maximum à l'intérieur de (D).

Il en résulte qu'une courbe de niveau $z =$ const. sera constituée par un certain nombre de courbes fermées. Lorsque z varie, ce nombre variera chaque fois que z passe par une des valeurs correspondant à un minimum ou à un col de la surface.

Soient maintenant z_1 et z_2 deux constantes comprises entre z_i et z_e. La courbe de niveau complète $z(x, y) = z_1$ se composera d'un certain nombre de courbes fermées. Si z_2 est plus grand et très voisin de z_1, la courbe de niveau $z(x, y) = z_2$ se composera d'un système de courbes fermées peu différentes de

celles constituant $z(x, y) = z_1$ et tendant uniformément vers ces dernières lorsque z_2 tend vers z_1.

Dans tout ce qui suit nous supposerons toujours que chacune des courbes fermées partielles du lieu géométrique $z(x, y) = z_2$ entoure complètement la courbe correspondante du lieu $z(x, y) = z_1$ à laquelle elle se réduit lorsque z_2 tend vers z_1.

Ceci exige, comme on peut aisément s'en rendre compte, que la fonction $z(x, y)$ prenne, sur les courbes C_i, une valeur inférieure à celle qu'elle prend sur les courbes de niveau voisines intérieures à (D). Il en résulte que z ne prend jamais, dans (D) de valeur inférieure à z_i et, comme il ne prend pas non plus de valeur supérieure à z_e, il suffira, comme nous l'avons fait plus haut, de considérer les valeurs de z comprises entre z_i et z_e.

Dans le cas particulier où C_e est entièrement située à l'infini, les courbes $z(x, y) = \text{const.} < z_e$ auront tous leurs points à distance finie.

On vérifie immédiatement que lorsque le domaine (D) est constitué simplement par l'aire limitée extérieurement par une courbe fermée C_e, les courbes C_i se réduisent à un ou plusieurs points où z est minimum et la condition restrictive d'enveloppement est toujours vérifiée.

Il résulte de ce qui précède que, lorsque z croît de z_i à z_e, les courbes de niveau $z = z(x, y)$ balayent le domaine (D) d'une manière continue et toujours dans le même sens. Si nous appelons $S(z)$ l'aire totale limitée extérieurement par la courbe $z(x, y) = z$, $S(z)$ sera donc une fonction continue croissante de z.

Appliquons à la région contenue entre $z(x, y) = z_i$ et $z(x, y) = z = \text{const.}$ la première formule de Green

$$(2) \qquad \iint (z_x^2 + z_y^2)\, dS + \iint z\Delta^2 z\, dS = \int_z z\partial z/\partial n\, ds - \int_{z_i} z\partial z/\partial n\, ds,$$

où ds et dS représentent respectivement l'élément d'arc et l'élément d'aire, et où $\partial z/\partial n$ désigne la normale extérieure. Les intégrales curvilignes sont prises le long des courbes de niveau limitant intérieurement et extérieurement le domaine d'intégration.

Si dn est la distance normale entre les courbes voisines z et $z + dz$, nous aurons évidemment

$$(3) \qquad dS/dz = \int_z dn/dz\, ds = \int_z ds/(z_x^2 + z_y^2)^{1/2} = \int_z ds/|\operatorname{grad} z|.$$

D'autre part, en appelant I l'intégrale

$$(4) \qquad I = \iint (z_x^2 + z_y^2)\, dS = \iint |\operatorname{grad} z|^2\, dS,$$

étendue au domaine limité par $z = z_i$ et $z = z$, nous pourrons évidemment écrire

$$(5) \qquad I = \int\int |\operatorname{grad} z|^2 \, dn/dz \, ds \, dz,$$

et par conséquent

$$(6) \qquad dI/dz = \int_z |\operatorname{grad} z| \, ds.$$

En appliquant maintenant au produit $(dS/dz)(dI/dz)$ l'inégalité de Schwarz, il vient

$$(7) \qquad (dS/dz)(dI/dz) \geqq \left[\int_z ds\right]^2 = L^2(z),$$

où $L(z)$ est la somme des longueurs des courbes fermées partielles constituant la courbe de niveau $z(x, y) = z$ complète. Si nous tenons compte de l'inégalité isopérimétrique $L^2 \geqq 4\pi S$, (7) a pour conséquence

$$(8) \qquad (dS/dz)(dI/dz) \geqq 4\pi S(z).$$

Mais nous pouvons d'autre part calculer directement la valeur de dI/dz. La relation (2), dérivée par rapport à z, donne en effet

$$(9) \qquad dI/dz = -d\left(\int_{z_i}^{z} z\Delta^2 z \, dS\right)/dz + d\left(\int_z z\partial z/\partial n \, ds\right)/dz.$$

La première de ces deux intégrales a pour valeur $-z\Delta^2 z \, dS/dz$. Pour calculer la seconde, remarquons que

$$(10) \qquad \int_z z\partial z/\partial n \, ds = z\int_{z_i}^{z} \Delta^2 z \, dS + z\int_{z_i} \partial z/\partial n \, ds,$$

d'où il résulte que

$$(11) \qquad \begin{aligned} dI/dz &= -z\Delta^2 z \, dS/dz + d\left[z\int_{z_i}^{z} \Delta^2 z \, dS + z\int_{z_i} \partial z/\partial n \, ds\right]/dz \\ &= \int_{z_i}^{z} \Delta^2 z \, dS + \int_{z_i} \partial z/\partial n \, ds. \end{aligned}$$

Remplaçons maintenant, dans (8), dI/dz par sa valeur (11) ci-dessus et prenons, au lieu de z, S comme variable indépendante. Ceci peut se faire sans ambiguité puisque, comme nous l'avons vu, entre z_i et z_e, S est une fonction toujours croissante de z. Donc dS/dz sera toujours positif, et nous pourrons écrire

$$(12) \qquad \int_{S_i}^{S} \Delta^2 z \, dS \geqq 4\pi S(dz/dS) - \int_{z_i} \partial z/\partial n \, ds,$$

d'où

$$(13) \qquad \int_{S_i}^{S} f(z) dS \geqq 4\pi S(dz/dS) - \int_{z_i} \partial z/\partial n \, ds,$$

et enfin

$$(14) \qquad dz/dS \leqq (1/4\pi S)\left[\int_{S_i}^{S} f(z)\,dS + \int_{z_i} \partial z/\partial n\,ds\right],$$

ce qui constitue l'inégalité fondamentale que nous désirions établir.

Si l'ensemble des courbes C_i se réduit à une série de points, nous aurons évidemment $S = 0$ pour $z = z_i$. On voit facilement que, dans ce cas, l'inégalité fondamentale devient

$$(15) \qquad dz/dS \leqq (1/4\pi S) \cdot \int_0^S f(z)\,dS,$$

et par conséquent, en particulier, puisque $f(z)$ est une fonction continue bornée

$$(16) \qquad \lim_{S=0} dz/dS \leqq (1/4\pi) \cdot f(z_i).$$

Nous avons admis jusqu'ici que la fonction $f(z)$ était une fonction positive de z. Si $f(z)$ est une fonction négative, nous pourrons écrire

$$(17) \qquad \Delta^2(-z) = -f(-(-z)),$$

de sorte que l'inégalité (14) s'appliquera à la fonction $-z$. Par conséquent

$$(18) \qquad -dz/dS \leqq -(1/4\pi S)\left[\int_{S_i}^{S} f(z)\,dS + \int_{z_i} \partial z/\partial n\,ds\right],$$

d'où

$$(19) \qquad dz/dS \geqq (1/4\pi S)\left[\int_{S_i}^{S} f(z)\,dS + \int_{z_i} \partial z/\partial n\,ds\right],$$

c'est-à-dire que le signe de l'inégalité devra être inversé.

Nous allons maintenant rechercher dans quel cas l'inégalité fondamentale devient une égalité. Pour que l'inégalité de Schwarz soit une égalité, il faut que $|\operatorname{grad} z|$ soit constant, c'est-à-dire que les courbes $z = \text{const.}$ soient parallèles. Pour que l'inégalité isopérimétrique soit une égalité, il faut que ces mêmes courbes soient des cercles. La condition nécessaire et suffisante pour que l'inégalité fondamentale devienne une égalité est donc que les courbes de niveau $z = \text{const.}$ constituent un système de cercles concentriques. On aura dans ce cas

$$(20) \qquad dz/dS = (1/4\pi S)\left[\int_{S_i}^{S} f(z)\,dS + \int_{z_i} \partial z/\partial n\,ds\right].$$

Multiplions les deux membres par $4\pi S$, puis dérivons par rapport à S. Il vient

$$(21) \qquad 4\pi d(S\,dz/dS)/dS = f(z).$$

Prenons comme variable indépendante le rayon r du cercle d'aire S, c'est-à-dire posons $S = \pi r^2$. Ceci donnera à l'équation (21) la forme suivante

$$(22) \qquad 1/r \cdot d(r\, dz/dr)/dr = f(z),$$

ce qui est, comme on pouvait s'y attendre, l'équation (1) écrite en coordonnées polaires et en supposant que z ne dépend que du rayon r.

Passons maintenant au cas de l'espace à trois dimensions. Nous considérerons un domaine (D) limité intérieurement par n surfaces fermées extérieures les unes aux autres $S_{i,1}, S_{i,2}, \cdots, S_{i,n}$, et extérieurement par une surface fermée S_e entourant complètement les surfaces S_i et n'ayant pas de points communs avec elles.

Les surfaces S_i et S_e seront constituées par un nombre fini de portions régulières. La surface S_e pourra, en particulier, être située entièrement à l'infini.

Nous considérerons ensuite une solution $u(x, y, z)$ de l'équation différentielle

$$(23) \qquad u_{xx} + u_{yy} + u_{zz} = \Delta^2 u = f(u)$$

finie et continue dans (D), qui prend la valeur constante u_i sur les surfaces S_i et la valeur constante $u_e > u_i$ sur la surface S_e. Nous admettrons que, lorsque $u_i \leqq u \leqq u_e$, $f(u)$ est une fonction analytique positive, bornée et continue de u.

La fonction $u(x, y, z)$ sera une fonction analytique dans tout le domaine (D), sauf peut-être sur les frontières S_i et S_e. Les surfaces de niveau $u(x, y, z) = \text{const.}$ contenues à l'intérieur de (D) seront donc constituées par un certain nombre de surfaces fermées, nombre que dépendra de la valeur particulière de u. Les dérivées partielles u_x, u_y et u_z restant toujours finies, les surfaces de niveau n'auront jamais de points communs. D'autre part, comme, par hypothèse, $f(u)$ est toujours positif, la fonction $u(x, y, z)$ ne pourra avoir de maximum à l'intérieur de (D).

Nous admettrons que, si u_1 et u_2 sont deux valeurs voisines de u, avec $u_2 > u_1$, chacune des surfaces fermées partielles constituant la surface de niveau $u = u_2$ entoure complètement la surface fermée correspondante de $u = u_1$ vers laquelle elle tend lorsque u_2 tend vers u_1.

Ceci exige que la fonction $u(x, y, z)$ prenne, sur les surfaces S_i, une valeur inférieure à celle qu'elle prend sur les surfaces de niveau voisines intérieures à (D). Il en résulte donc que $u(x, y, z)$ sera toujours compris entre u_i et u_e. En particulier, si S_e est située entièrement à l'infini, les surfaces de niveau $u(x, y, z) = \text{const.} < u_e$ auront tous leurs points à distance finie.

Lorsque (d) se réduit au volume limité extérieurement par une surface fermée régulière, les surfaces S_i sont constituées par le point ou les points

où $u(x,y,z)$ prend sa valeur minimum. On vérifie alors immédiatement que la condition d'enveloppement est toujours satisfaite.

Il résulte de ce qui précède que, lorsque u croît de u_i à u_e, les surfaces $u(x,y,z)=u$ balayent le domaine (D) d'une manière continue et toujours dans le même sens. Si nous appelons $V(u)$ le volume total limité extérieurement par la surface de niveau $u(x,y,z)=u$, V sera donc une fonction continue croissante de u.

L'inégalité fondamentale s'établira à peu près exactement comme dans le cas de deux dimensions. La première formule de Green, appliquée à la région comprise entre $u(x,y,z)=u_i$ et $u(x,y,z)=u=\text{const.}$, donnera tout d'abord

$$\text{(24)} \qquad \iiint (u_x^2+u_y^2+u_z^2)\,dV + \iiint u\Delta^2 u\,dV = \iint_{u_i} u\partial u/\partial n\,dS - \iint_{u} u\partial u/\partial n\,dS,$$

où dV et dS représentent respectivement l'élément de volume et l'élément de surface, et où $\partial u/\partial n$ désigne la dérivée normale extérieure. Les intégrales doubles sont prises le long des surfaces de niveau limitant intérieurement et extérieurement le domaine d'intégration.

En appelant I l'intégrale

$$\text{(25)} \qquad I=\iiint (u_x^2+u_y^2+u_z^2)\,dV,$$

étendue à la même région, on démontrera ensuite que, en vertu de l'inégalité de Schwarz

$$\text{(26)} \qquad (dV/du)(dI/du) \geqq S^2(u),$$

où $S(u)$ représente la somme des aires des surfaces fermées constituant la surface de niveau $u(x,y,z)=u$. Si l'on applique maintenant l'inégalité isopérimétrique $S(u)\geqq[36\pi V^2(u)]^{1/3}$, il vient

$$\text{(27)} \qquad (dV/du)(dI/du) \geqq [36\pi V^2]^{2/3}.$$

On établira d'autre part que

$$\text{(28)} \qquad dI/du=\int_{V_i}^{V} \Delta^2 u\,dV + \iint_{S_i} \partial u/\partial n\,dS,$$

et par conséquent, enfin, que

$$\text{(29)} \qquad [36\pi V^2]^{2/3} du/dV \leqq \int_{V_i}^{V} f(u)\,dv + \iint_{S_i} \partial u/\partial n\,dS,$$

ce qui constitue l'inégalité fondamentale que nous voulions démontrer, dans le cas de trois dimensions.

On voit immédiatement que des inégalités analogues pourraient être obtenues de la même manière dans le cas d'un nombre supérieur de dimensions.

On démontrera comme plus haut que, pour que l'inégalité fondamentale devienne une égalité, il est nécessaire et suffisant que les surfaces $u(x, y, z) =$ const. constituent un système de sphères concentriques. Nous aurons donc, dans ce dernier cas

$$(30) \qquad [36\pi V^2]^{1/3} du/dV = \int_{V_i}^{V} f(u)\,dV + \iint_{S_i} \partial u/\partial n\, dS.$$

Posons $V = 4/3\pi r^3$, puis dérivons les deux membres par rapport à r. Il vient

$$(31) \qquad 1/r^2 \cdot d(r^2 du/dr)/dr = f(u),$$

ce qui n'est autre que l'équation (23) écrite en coordonnées polaires et en admettant que u est seulement fonction de r.

Si la surface S_i se réduit à une série de points, on aura évidemment $V = 0$ pour $u = u_i$. L'inégalité (29) prend alors la forme

$$(32) \qquad [36\pi V^2]^{1/3} du/dV \leqq \int_0^V f(u)\,dV,$$

d'où l'on déduit que

$$(33) \qquad \lim_{V=0} V^{1/3} du/dV \leqq (36\pi)^{-1/3} f(u_i).$$

On démontrera enfin facilement que si $f(u)$, au lieu d'être une fonction positive, est une fonction toujours négative, il faudra changer le sens de l'inégalité fondamentale qui deviendra ainsi

$$(34) \qquad (36\pi V^2)^{1/3} du/dV \geqq \int_{V_i}^{V} f(u)\,dV + \iint_{S_i} \partial u/\partial n\, dS.$$

3. Interprétation géométrique de l'inégalité fondamentale. L'inégalité fondamentale est susceptible d'une interprétation géométrique intéressante.

Admettons que, dans le cas de deux dimensions, $f(z)$ est une fonction croissante de z et que, pour $z = z_i$, l'aire $S(z_i)$ des courbes C_i et l'intégrale $\int_{z_i} (\partial z/\partial n)\,ds$ étendue à ces mêmes courbes aient une valeur fixe, indépendante du système particulier de courbes C_i considéré.

La courbe $z = z(S)$ passera alors toujours par le point fixe $z = z_i$, $S = S_i$, et aura son coefficient angulaire, en tout point, inférieur à la quantité

$$(4\pi S)^{-1}\Big[\int_{S_i}^{S} f(z)\,dS + \int_{z_i} \partial z/\partial n\, ds\Big].$$

Appelons $Z(S)$ la solution de l'équation (20) qui passe également par le point z_i, S_i. Nous aurons alors

$$dZ/dS - dz/dS \geqq \int_{S_i}^{S} (f(Z) - f(z))\,dS. \tag{35}$$

Or, aux environs immédiats du point z_i, S_i, on a évidemment $d(Z-z)/dS \geqq 0$, d'où $Z \geqq z$ et $f(Z) \geqq f(z)$. En raisonnant de proche en proche, on voit qu'on aura toujours $d(Z-z)/dS \geqq 0$ et $Z \geqq z$. Il en résulte que la courbe $Z(S)$ sera toujours située au dessus de la courbe $z(S)$ et que l'écart entre ces deux courbes augmentera en même temps que S.

La courbe $Z(S)$ fournit donc, pour chaque valeur de z, une limite inférieure pour l'aire limitée par la courbe de niveau $z(x,y) = z$.

Dans le cas de trois dimensions, on peut établir une propriété analogue.

Nous admettrons encore une fois que $f(u)$ est une fonction toujours positive et croissante de u, et que le volume V_i limité par les surfaces S_i ainsi que l'intégrale $\int\int_{S_i} \partial u/\partial n\, dS$ ont une valeur fixe, indépendante du système particulier de surfaces S_i. Pour $u = u_i$, on aura donc toujours $V = V_i$. Appelons $U(V)$ la solution de l'équation (30) passant également par le point $u = u_i$, $V = V_i$. On aura évidemment

$$dU/dV - du/dV \geqq (36\pi V^2)^{-2/3} \int_{V_i}^{V} (f(U) - f(u))\,dV, \tag{36}$$

et par conséquent la courbe $U(V)$ se trouvera toujours au dessus de la courbe $u(V)$. La courbe $U(V)$ permettra donc, pour toutes les valeurs de u, d'obtenir une limite inférieure du volume V limité par la surface $u(x,y,z) = u$.

Nous allons maintenant appliquer les inégalités fondamentales (14) et (29) à une série de cas particuliers correspondant à différentes formes de la fonction f.

4. Condensateur électrostatique de capacité minimum. Considérons un condensateur constitué par deux surfaces conductrices fermées, telles que la surface extérieure S_e entoure complètement, sans la toucher, la surface intérieure S_i. Ces deux surfaces sont portées respectivement aux potentiels u_e et u_i, avec $u_e > u_i$.

Dans le domaine compris entre les deux surfaces, le potentiel $u(x,y,z)$ vérifie l'équation de Laplace $\Delta^2 u = 0$, et l'intégrale $\int \partial u/\partial n\, dS$ prise sur l'une quelconque des surfaces équipotentielles comprises entre S_i et S_e a une valeur constante et égale à $4\pi Q$, où Q est la charge de l'armature intérieure.

On voit immédiatement que les conditions d'application de l'inégalité

(28) sont remplies, et celle-ci devient, dans le cas actuel, en posant $V = 4\pi r^3/3$, l'inégalité $r^2 du/dr \leqq Q$. Divisons les deux membres de cette inégalité par r^2 et intégrons ensuite entre r_i et r, avec bien entendu, $V_i = 4\pi r_i^3/3$. En tenant compte de ce que, pour $u = u_i$, on a $r = r_i$, il vient $u \leqq u_i + Q(1/r_i - 1/r)$, et en particulier, sur l'armature extérieure, où $r = r_e$, $u_e \leqq u_i + Q(1/r_i - 1/r_e)$. En appelant C la capacité du condensateur, ceci peut encore s'écrire

$$C = Q/(u_e - u_i) \geqq 1/(1/r_i - 1/r_e) = (3/4\pi)^{1/3}/(V_i^{-1/3} - V_e^{-1/3}),$$

c'est-à-dire que:

Parmi tous les condensateurs électrostatiques tels que les armatures extérieures et intérieures entourent des volumes donnés, celui constitué par deux sphères concentriques a la plus petite capacité.

On peut évidemment faire un raisonnement analogue dans le cas d'un condensateur électrostatique constitué par deux armatures cylindriques parallèles telles que l'une entoure complètement l'autre et dont les sections droites ont des aires données. La capacité par unité de longueur minimum correspond alors au condensateur constitué par deux cylindres circulaires coaxiaux.

Dans le cas du condensateur de capacité minimum, notre méthode est évidemment identique à celle utilisée par G. Szegö dans le travail déja mentionné plus haut.

5. Application à la théorie des membranes. La déformation normale d'une membrane plane uniforme et uniformément tendue soumise à une pression constante est donnée par l'équation $\Delta^2 z = -p/T$, où T est la tension et p la pression par unité de surface.

Supposons la membrane limitée par un contour fermé sans points multiples. Sur ce contour on devra avoir $z = 0$. Les courbes de niveau $z =$ const. seront également des courbes fermées qui entourent le point ou les points où z est maximum. Prenons ces points pour origine des S et posons $z = z_M - z_1$, où z_M représente la valeur maximum de z.

La quantité z_1 vérifiera l'équation $\Delta^2 z_1 = p/T$, et on voit sans peine que l'inégalité fondamentale pourra être appliquée au système des courbes $z_1 =$ const. Dans le cas présent, (14) devient simplement $dz_1/dS \leqq p/4\pi T$, d'où, en intégrant entre $z_1 = 0$ et $z_1 = z_M$, $z_M \leqq pS_e/4\pi T$, c'est-à-dire que:

Parmi toutes les membranes de même aire, soumises à la même tension, et supportant la même pression uniforme, la membrane circulaire présente la plus grande déformation normale maximum.

6. Application à la théorie de la torsion des prismes. L'étude de la torsion d'un prisme droit de section quelconque se ramène, comme on le sait, à la détermination de la solution de l'équation $\Delta^2 z = -2$, qui s'annule sur le contour de la section droite. Nous supposerons celle-ci limitée par une courbe fermée sans points multiples. Nous pourrons appliquer les résultats du paragraphe précédent et, en posant encore une fois $z = z_M - z_1$, écrire

$$dz_1/dS \leqq 1/2\pi. \tag{37}$$

L'angle de rotation du prisme, par unité de longueur, a pour valeur $\vartheta = M/C$, où M est le couple de torsion et C le module de torsion donné par $C = 2G \int z\, dS$, G étant le module de glissement et l'intégrale étant étendue à l'aire de la section droite. On vérifie aisément qu'on a également $C = 2G \int S\, dz_1$, et par conséquent, en vertu de (37)

$$C = 2G \int S dz_1 \leqq 2G \int_0^{S_e} S\, dS/2\pi = GS_e^2/2\pi, \tag{38}$$

S_e étant ici l'aire de la section droite. Ceci montre que:

Parmi tous les prismes de même nature et dont la section droite a la même aire, le prisme de section circulaire possède le module de torsion le plus élevé.

7. Application à l'hydrodynamique des liquides visqueux. On démontre aisément, par un calcul identique à celui qui précède, que le débit d'un liquide visqueux s'écoulant en régime laminaire dans un tube cylindrique de section S_0 est soumis à l'inégalité suivante: $Q \leqq (P\rho/8\pi\mu L) S_0^2$, où Q est le débit, P la différence de pression aux extrémités du tube, ρ la densité, μ le coefficient de viscosité. L'égalité n'a lieu que pour une section circulaire.

Donc, si l'on fait écouler, en régime laminaire et sous l'influence d'une même différence de pression par unité de longueur, un même liquide visqueux dans des tubes cylindriques de même longueur, on obtiendra le débit maximum avec un tube de section circulaire.

8. Problème de Rayleigh. La fonction caractéristique fondamentale de l'équation des membranes vibrantes

$$\Delta^2 z + k^2 z = 0, \tag{39}$$

relativement à une courbe fermée (C) sans points multiples est la solution $z(x, y)$ de cette équation, toujours positive, nulle sur le contour. La valeur

correspondante de k est, à un facteur constant près, la fréquence fondamentale.

D'après les propriétés de l'équation (39), la fonction $z(x,y)$ ne peut pas posséder de minimum à l'intérieur de la courbe (C), mais elle peut posséder plusieurs maxima distincts. Appelons z_M la plus grande valeur de $z(x,y)$ dans (C), cette valeur pouvant être atteinte en plusieurs points distincts. Les courbes de niveau $z = \text{const.}$, lorsque z varie entre 0 et z_M, forment un système de courbes fermées analytiques sans points communs s'enveloppant mutuellement. Posons $z_1 = z_M - z$. La fonction $z_1(x,y)$ vérifiera l'équation $\Delta^2 z_1 = k^2(z_M - z_1)$, dont le second membre sera toujours positif. Nous pourrons donc appliquer l'inégalité fondamentale (15) qui deviendra dans le cas actuel

$$dz_1/dS \leqq (k^2/4\pi S) \int_0^S (z_M - z_1)\, dS. \tag{40}$$

Si S_0 représente l'aire de la membrane, lorsque S variera entre S et S_0, z_1 sera une fonction croissante de S et dz_1/dS sera positif. En multipliant les deux membres de (40) par $S\, dz_1/dS$ et en intégrant entre 0 et S_0, on aura donc

$$\int_0^{S_0} S(dz_1/dS)^2 dS \leqq (k^2/4\pi) \int_0^{S_0} dz_1/dS \left[\int_0^S (z_M - z_1)\, dS \right] dS.$$

Le second membre peut être intégré par parties et devient

$$\int_0^{S_0} dz_1/dS \left[\int_0^S (z_M - z_1)\, dS \right] dS = \left[z_1 \int_0^S (z_M - z_1)\, dS \right]_0^{S_0} - \int_0^{S_0} z_1(z_M - z_1) dS$$

$= \int_0^{S_0} (z_M - z_1)^2 dS$ de sorte que, en revenant à la variable z, on obtient

$$\int_0^{S_0} S(dz/dS)^2 dS \leqq (k^2/4\pi) \int_0^{S_0} z^2 dS.$$

Posons maintenant $S = \pi r^2$. Il vient

$$\int_0^{r_0} (dz/dr)^2 r\, dr \leqq k^2 \int_0^{r_0} z^2 r\, dr, \tag{41}$$

avec $S_0 = \pi r_0^2$. Or $z(r)$ est une fonction continue et dérivable de r, pour $0 \leqq r \leqq r_0$, qui s'annule pour $r = r_0$. On pourra donc la développer en série convergente et dérivable de fonctions de Bessel $J_0(\lambda_n r)$, où les constantes λ_n sont définies par la condition

$$J_0(\lambda_n r_0) = 0. \tag{42}$$

On aura donc $z = \Sigma_n A_n J_0(\lambda_n r)$, où les A_n sont des constantes. En introduisant cette valeur de z dans l'inégalité (41), celle-ci devient

$$\frac{\int_0^{r_0}(dz/dr)^2 r\,dr}{\int_0^{r_0} z^2 r\,dr} = \frac{\Sigma A_n{}^2\lambda_n{}^2 \int_0^{r_0} rJ_0{}^2(\lambda_n r)\,dr}{\Sigma A_\kappa{}^2 \int_0^{r_0} rJ_0{}^2(\lambda_n r)\,dr} \geqq k^2.$$

Les quantités λ_n forment une suite de nombres croissant indéfiniment. En appelant $\lambda_0 r_0$ la plus petite des racines de (42), on aura donc $k^2 \geqq \lambda_0{}^2 = (2{,}405\cdots)^2\pi/S_0$ $(\lambda_0 r_0 = 2{,}405\cdots)$ c'est-à-dire que la fréquence fondamentale d'une membrane non circulaire sera toujours supérieure à la fréquence fondamentale de la membrane circulaire de même aire.

On peut démontrer de la même manière la propriété correspondante dans le cas de trois dimensions.

9. Problème de Liapounoff. Considérons une masse fluide incompressible, immobile, de volume V_0, en équilibre sous l'influence de son attraction newtonienne propre. On sait que, dans ces conditions, la surface extérieure sera une surface équipotentielle.

Nous devrons considérer séparément le potentiel newtonien u_i à l'intérieur de la masse et le potentiel u_e à l'extérieur de celle-ci. Ces deux fonctions vérifient les équations suivantes $\Delta^2 u_i = -4\pi k\rho$, $\Delta^2 u_e = 0$, où ρ est la densité (constante) du fluide et k la constante de gravitation. A la surface même du liquide, on devra évidemment avoir $u_i = u_e = u_0$.

On voit immédiatement que, à l'extérieur de la masse fluide, l'inégalité fondamentale (33) pourra être appliquée aux surfaces équipotentielles $u_e = \text{const.}$ Par conséquent

$$(36\pi V^2)^{1/3} du_e/dV \geqq \int \partial u_e/\partial n\, dS = -4\pi k\rho V_0.$$

En posant $V = 4\pi r^3/3$, ceci peut encore s'écrire $du_e/dr + V_0 k\rho/r^2 \geqq 0$. Intégrons le premier membre de cette inégalité entre r et l'infini. Pour V très grand, les surfaces équipotentielles diffèrent très peu de sphères, de sorte que, dans ce cas $u_e = V_0 k\rho/r +$ termes en $r^{-2}, r^{-3}, \cdots$ et par conséquent $u_e - V_0 k\rho/r$ s'annule pour $r = \infty$. Il reste alors $-(u_e - V_0 k\rho/r) \geqq 0$, ou bien

$$u_e \leqq V_0 k\rho/r. \tag{43}$$

Considérons ensuite le potentiel u_i à l'intérieur de la masse fluide. Celui-ci prendra évidemment une valeur maximum en un ou plusieurs points, mais ne possèdera certainement pas de minimum. Les surfaces équipotentielles, coincidant avec les surfaces d'égale pression, s'enveloppent mutuellement et

nous pourrons appliquer l'inégalité fondamentale (33) en prenant pour origine $V=0$ le point ou les points où u_i est maximum. Nous obtenons ainsi $(36\pi V^2)^{2/3} du_i/dV \geqq -4\pi k\rho V$, ce qui peut encore s'écrire $du_i/dr + 4/3 \cdot \pi k\rho r \geqq 0$, en posant comme plus haut, $V = 4\pi r^3/3$. Intégrons cette inégalité entre r et r_0. Il vient

$$u_0 - u_i + 2/3 \cdot \pi k\rho r_0^2 - 2/3 \cdot \pi k\rho r^2 \geqq 0\,,$$

d'où

$$u_i \leqq u_0 + 2/3 \cdot \pi k\rho r_0^2 - 2/3 \cdot \pi k\rho r^2,$$

et, en tenant compte de (43), $u_i \leqq 2\pi k\rho r_0^2 - 2/3 \cdot \pi k\rho r^2$. Or l'énergie potentielle de la masse a pour valeur

$$E = \tfrac{1}{2}\rho \int_0^{V_0} u_i dV = 2\pi\rho \int_0^{r_0} u_i r^2 dr,$$

et par conséquent, enfin

$$E \leqq 16/15 \cdot \pi^2 k\rho^2 r_0^5 = 16/15 \cdot \pi^2 k\rho^2 (3V_0/4\pi)^{5/3}.$$

Si donc il existe une forme d'équilibre non sphérique, son énergie potentielle gravitationnelle sera inférieure à celle de la forme sphérique d'équilibre de même volume.

Il en résulte donc que la sphère constitue la forme d'équilibre stable d'une masse fluide incompressible immobile.

Considérons maintenant une masse fluide constituée par des volumes V_1 et V_2 de deux liquides immiscibles différents, de densités constantes ρ_1 et ρ_2, avec $\rho_1 > \rho_2$. A l'état d'équilibre, on sait que les surfaces équipotentielles coincident avec les surfaces d'égale pression et les surfaces d'égale densité. D'autre part, pour que l'équilibre soit stable, il faut évidemment que la pression augmente au fur et à mesure que l'on s'éloigne de la surface extérieure. Nous ne considérerons que les formes d'équilibre où le liquide le moins dense entoure complètement le liquide le plus dense. La surface de séparation, de même que la surface extérieure, seront des surfaces équipotentielles.

Le potentiel newtonien de la masse totale sera une fonction continue et à dérivées partielles continues des coordonnées. Ce sera donc aussi une fonction continue et à dérivée continue du volume V limité par la surface équipotentielle $u(x, y, z) = \text{const.}$ On aura de plus:

Pour $V > V_1 + V_2$, $\Delta^2 u = 0$; pour $V_1 + V_2 > V > V_1$, $\Delta^2 u = -4\pi k\rho_2$; et pour $V_1 > V$, $\Delta^2 u = -4\pi k\rho_1$.

En introduisant ces valeurs dans l'inégalité fondamentale et en procédant comme ci-dessus, on démontrera aisément qu'une configuration d'équilibre éventuelle, différente de celle constituée par une sphère de densité ρ_1 et de

volume V_1, entourée d'une couche sphérique concentrique de densité ρ_2 et de volume V_2 possèdera une énergie potentielle inférieure à celle de la configuration sphérique concentrique. Celle-ci constituera donc certainement la forme d'équilibre stable de la masse.

Nous pourrons étudier de la même manière la forme d'équilibre d'une masse hétérogène constituée par des volumes $V_1, V_2, \cdots, V_n$ de n fluides incompressibles immiscibles de densités $\rho_1, \rho_2, \cdots, \rho_n$. Si nous supposons que $\rho_1 > \rho_2 \cdots > \rho_n$, nous pourrons, en utilisant toujours le même procédé, démontrer que la configuration d'équilibre stable sera constituée par une sphère de volume V_1 et de densité ρ_1, entourée d'une série de couches sphériques concentriques de densités décroissantes ayant respectivement les volumes $V_2, \cdots, V_n$ et les densités $\rho_2, \cdots, \rho_n$.

Ce dernier résultat peut aisément être généralisé au cas d'une variation continue de la densité.

Les résultats qui précèdent ont été obtenus d'une manière entièrement différente par L. Lichtenstein [11].

10. Inégalités relatives à la fonction de Green. Considérons, dans le plan, un contour fermé sans points multiples (C) et soit $P(a, b)$ un point intérieur quelconque fixe. La fonction de Green correspondante $G(a, b, x, y) = G(P, Q)$ s'annulera sur le contour et aura la forme suivante: $G(a, b, x, y) = \log \cdot r_{PQ} + H(a, b, x, y)$, où r_{PQ} est la distance entre les points $P(a, b)$ et $Q(x, y)$, et H est une fonction harmonique de x et de y. La fonction $G(P, Q)$ vérifie évidemment l'équation $\Delta^2 G = 0$, lorsque $Q \neq P$. Les courbes $G(P, Q) = \text{const.} = G$ sont des courbes analytiques, (sauf peut-être pour $G = 0$) fermées s'enveloppant mutuellement. Pour $G = -\infty$, la courbe se réduit au point P, tandis que pour $G = 0$, elle coincide avec (C). L'intégrale curviligne $\int \partial G/\partial n \, ds$ prise le long d'une des courbes fermées $G = \text{const.} > -\infty$ a une valeur constante et égale à 2π. It est donc possible d'appliquer l'inégalité fondamentale au système des courbes $G = \text{const.}$ et on obtient ainsi $dG/dS \leqq 1/2S$. Comme G croît en même temps que S, la dérivée sera positive et nous pourrons intégrer les deux membres de l'inégalité entre S et S_0, S_0 étant l'aire limitée par la courbe (C). Puisque, pour $G = 0$, on a $S = S_0$, on obtient ainsi $-G \leqq \frac{1}{2} \log S_0/S$, ce qu'on peut encore écrire $S \leqq S_0 \exp 2G$.

L'aire limitée par la courbe $G(x, y) = \text{const.}$ sera donc inférieure à $S_0 \exp. 2G$, sauf évidemment si (C) est un cercle dont P est le centre.

Dans le cas de trois dimensions, nous aurons à considérer le domaine limité par une surface régulière fermée (S). La fonction de Green aura la forme $G(P, Q) = -1/r_{PQ} + H(P, Q)$, où r_{PQ} est la distance du point fixe

$P(a, b)$ au point variable $Q(x, y)$ et H une fonction harmonique, et s'annulera sur la surface (S). L'intégrale de surface $\int\int (\partial G/\partial n)\,dS$ prise sur l'une quelconque des surfaces fermées analytiques $G = \text{const.} > -\infty$, aura la valeur constante 4π.

Il sera de nouveau possible d'appliquer l'inégalité fondamentale qui donnera maintenant $dG/dV \leqq 4\pi(36\pi V^2)^{-1/3}$. En intégrant entre V et V_0, où V_0 est le volume limité par la surface (S), il viendra $G \geqq (4\pi/3)^{1/3}(V_0^{-1/3} - V^{-1/3})$, puisque $V = V_0$ pour $G = 0$. L'inégalité ne devient une égalité que lorsque (S) est une sphère de centre P (sauf bien entendu pour les valeurs exceptionnelles $G = 0$ et $G = -\infty$).

D'autres inégalités relatives à la fonction de Green dans le plan peuvent être obtenues par simple changement de forme des résultats donnés plus haut.

On vérifie facilement que la solution de l'équation $\Delta^2 z = -1$ régulière à l'intérieur du contour (C) et s'annulant sur celui-ci est donnée par

$$z(x, y) = 1/2\pi \int\int G(x, y, \xi, \eta)\, d\xi\, d\eta,$$

de sorte que l'inégalité $z_M \leqq pS_e/4\pi T$, trouvée à la fin du § **5**, peut s'écrire

$$\int\int G(x, y, \xi, \eta)\, d\xi\, d\eta \leqq \tfrac{1}{2} S_0. \tag{44}$$

De même, l'inégalité (38) peut se mettre sous la forme suivante

$$\int\int\int\int G(x, y, \xi, \eta)\, dx dy d\xi d\eta \leqq \tfrac{1}{4} S_0^2.$$

Dans ses travaux relatifs à la résolution du problème de Dirichlet pour les équations de la forme $\Delta^2 u = c(x, y) u$ et $\Delta^2 u = F(x, y, u)$, E. Picard [12] a rencontré l'inégalité suivante

$$1/2\pi \int\int G(x, y, \xi, \eta)\, d\xi d\eta \leqq \lambda,$$

et a démontré que λ tendait vers zéro en même temps que l'aire limitée par le contour. Notre formule (44) donne pour λ la valeur exacte $S_0/4\pi$. Il n'y a aucune difficulté à obtenir des résultats analogues dans le cas de trois dimensions. On trouve ainsi

$$\int\int\int G(x, y, z, \xi, \eta, \zeta)\, d\xi d\eta d\zeta \leqq 1/6 \cdot (3V_0/4\pi)^{2/3},$$

$$\int\int\int\int\int\int G(x, y, z, \xi, \eta, \zeta)\, dx dy dz d\xi d\eta d\zeta \leqq 4\pi/45 \cdot (3V_0/4\pi)^{5/3}.$$

Encore une fois, ces inégalités ne deviennent des égalités que lorsque la frontière est un cercle ou une sphère.

Note additionelle.[1] Un examinateur nous a indiqué une démonstration de l'inégalité (12) beaucoup plus simple que celle donnée plus haut. En utilisant la même notation que dans le paragraphe **2**, on voit aisément que

[1] Received April 28, 1950.

$$\int_z \partial z/\partial n\, ds \times \int_z ds/(\partial z/\partial n) > [\int_z ds]^2 = L^2(z) > 4\pi S(z).$$

Mais $\int_z ds/(\partial z/\partial n) = dS/dz$, de sorte que $\int_z \partial z/\partial n\, ds \times dS/dz \geqq 4\pi S(z)$,

ou, en prenant S comme variable indépendante, $\int_z \partial z/\partial n\, ds \geqq 4\pi S.dz/dS$.

Si maintenant nous faisons usage de la formule de Green

$$\iint \Delta^2 z\, dS = \int_z (\partial z/\partial n)\, ds - \int_{z_1} \partial z/\partial n\, ds,$$

nous obtenons finalement l'inégalité (12),

$$\iint \Delta^2 z\, dS + \int_{z_1} \partial z/\partial n\, ds \geqq 4\pi S\, dz/dS.$$

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ON DENUMERABLY INDEPENDENT FAMILIES OF BOREL FIELDS.*

By S. Sherman.

In a posthumous paper [B] Banach showed that if one has a denumerably independent family of Borel fields (each with a measure) of subsets of a set Ω, then there exists a common extension of the measures to the Borel field generated by the Borel fields of the family such that the fields are stochastically independent. Since his proof is quite long we give a shorter proof which has the advantage that it displays the relation between a denumerably independent family of Borel fields and an isomorphic family of Borel fields in a product space. As a consequence of this the existence (and uniqueness) of Banach's extension measure is equivalent to the existence (and uniqueness) of an independent product measure, a solved problem.[1,2]

Let Ω be a set and let $\mathfrak{B}^\gamma = \{B^\gamma\}$ be a Borel field of subsets of Ω for each $\gamma \varepsilon \Gamma$. Let $\mathfrak{A} = \{\mathfrak{B}^\gamma \mid \gamma \varepsilon \Gamma\}$. We say that $E[H]$ is a $\mathfrak{A}$-rectangle [$\mathfrak{A}$-block] if $E[H] = \bigcap_\gamma B^\gamma$, where all except a finite [countable] collection of the B^γ are equal to Ω. In the sequel H (with or without subscripts) denotes an $\mathfrak{A}$-block. Let $\mathfrak{E}$ [$\mathfrak{B}$] be the [Borel] field generated by the rectangles. We note that i) if $A \varepsilon \mathfrak{E}$ then A is a finite disjoint union of rectangles, ii) if E is an $\mathfrak{A}$-rectangle then E is an $\mathfrak{A}$-block, and iii) $\mathfrak{B}$ is the smallest Borel field containing each $\mathfrak{B}^\gamma \varepsilon \mathfrak{A}$.

Lemma. *If $A \varepsilon \mathfrak{E}$, then*

*) $p \varepsilon A \Longrightarrow$ for some $\underline{H} \subset A$, $p \varepsilon \underline{H}$,

**) $p \varepsilon A' \Longrightarrow$ for some $\bar{H} \subset A'$, $p \varepsilon \bar{H}$.

(Note: A' means the complement of A.)

Proof. Trivial since both A and A' are disjoint unions of $\mathfrak{A}$-rectangles.

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* Received October 24, 1949; revised January 20, 1950.

[1] See [K] where a recent proof as well as references to the literature are given. See also [SJ].

[2] According to Mr. Henry Helson another (different) proof by Saks of Banach's theorem is to appear in a Polish Journal.

THEOREM. *If $A \varepsilon \mathfrak{B}$, then * and **.*

Proof. If A satisfies * and **, then A' satisfies * and **. Let $A = \bigcap_{\alpha=1}^{\infty} B_\alpha$, where for each α,

$$p \varepsilon B_\alpha \Longrightarrow \text{ for some } \underline{H}_\alpha \subset B_\alpha,\ p \varepsilon \underline{H}_\alpha,$$

$$p \varepsilon B'_\alpha \Longrightarrow \text{ for some } \bar{H}_\alpha \subset B'_\alpha,\ p \varepsilon \bar{H}_\alpha.$$

If $q \varepsilon A$, then $q \varepsilon \bigcap_{\alpha=1}^{\infty} \underline{H}_\alpha$, which is an $\mathfrak{A}$-block. If $q \varepsilon A'$, then for some $\bar{\alpha}$, $p \varepsilon \bar{H}_{\bar{\alpha}}$. which is an $\mathfrak{A}$-block. Thus A satisfies * and **. We have shown that the sets satisfying * and ** form a Borel field including $\mathfrak{E}$. Since $\mathfrak{B}$ is included in any Borel field including $\mathfrak{E}$, every set in $\mathfrak{B}$ satisfies * and **.

COROLLARY. *If $B \varepsilon \mathfrak{B}$, then B is the union (possibly non-countable) of $\mathfrak{A}$-blocks in B.*

We now introduce some notation associated with product spaces, largely following Kakutani [K]. Let $\Omega^* =$ the set of all Γ-sequences, $\omega^* = \{\omega^\gamma \mid \gamma \varepsilon \Gamma\}$ with $\omega^\gamma \varepsilon \Omega$ for each $\gamma \varepsilon \Gamma$. Let $B^{*\gamma} = \{\omega^* \mid \omega^\gamma \varepsilon B^\gamma\}$, $\mathfrak{B}^{*\gamma} = \{B^{*\gamma}\}$, $\mathfrak{A}^* = \{\mathfrak{B}^{*\gamma} \mid \gamma \varepsilon \Gamma\}$. Note: $\mathfrak{A}^*$ is a denumerably independent family of Borel fields of subsets of Ω^*. Let $E^*[H^*]$ represent an $\mathfrak{A}^*$-rectangle [block] and let $\mathfrak{E}^*[\mathfrak{B}^*]$ represent the [Borel] field generated by the $\mathfrak{A}^*$-rectangles. The previous lemma, theorem, and corollary apply to family $\mathfrak{A}^*$.

If with each $\mathfrak{B}^{*\gamma}$ we have m^γ a countably additive measure (all measures are countably additive in the sequel), where $m^\gamma(\Omega^*) = 1$, then it is known, (e. g. [K]) that there is a unique measure m^* on $\mathfrak{B}^*$ such that for each $\mathfrak{A}^*$-block $H^* = \bigcap B^{*\gamma}$, $m^*(H^*) = \prod m^\gamma(B^\gamma)$. We are now in a position to prove the

THEOREM (BANACH). *If $\mathfrak{A}$ is a denumerably independent family $\{\mathfrak{B}^\gamma \mid \gamma \varepsilon \Gamma\}$ of Borel fields of subsets of Ω and for each $\mathfrak{B}^\gamma$, $\gamma \varepsilon \Gamma$, there is a measure m^γ such that $m^\gamma(\Omega) = 1$, then there is a unique measure m on $\mathfrak{B}$ the Borel field generated by $\mathfrak{A}$ such that*

$$A \varepsilon \mathfrak{B}^\gamma \Longrightarrow m(A) = m^\gamma(A),$$

$$\text{(stochastic independence) } \mathfrak{A}\text{-block } H = \bigcap_\gamma B^\gamma \Longrightarrow mH = \prod_\gamma m^\gamma(B^\gamma).$$

Proof. Let ϕ be the transformation from Ω into Ω^* defined by $\phi(\omega) = \{\omega^\gamma \mid \gamma \varepsilon \Gamma\}$ where $\omega^\gamma = \omega$ for every γ; let D^* be the range of ϕ.

Banach's theorem can be proved by writing $m\phi^{-1}(A^*) = m^*A^*$—all that needs proof is that ϕ^{-1} (which is obviously a σ-homomorphism from $\mathfrak{B}^*$ onto $\mathfrak{B}$) is an isomorphism. For this in turn it is sufficient to prove that if $A^* \varepsilon \mathfrak{B}^*$ and $A^* \cap D^* = \Lambda$ then $A^* = \Lambda$. If A^* is an $\mathfrak{A}^*$ block, $A^* = \bigcap_\gamma B^{*\gamma}$, $B^{*\gamma} = \{\omega^* \mid \omega^\gamma \varepsilon B^\gamma\}$, then $\phi^{-1}(A^*) = \bigcap_\gamma B^\gamma$ and the desired conclusion follows from the assumed denumerable independence; the general case is now a consequence of the corollary.[3] Note that the σ-isomorphism between $\mathfrak{B}^*$ and $\mathfrak{B}$ is a set-theoretical proposition independent of measure assumptions.

INSTITUTE FOR ADVANCED STUDY.

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[3] The method for proving the σ-isomorphism between $\mathfrak{B}^*$ and $\mathfrak{B}$ is the referee's and effects a considerable simplification of the author's method, which was long and roundabout.

CERTAIN CLASSES OF SERIES TO SERIES TRANSFORMATION MATRICES.*

By P. Vermes.

1. In a recent paper [1] there were described series to series transformation matrices having the property that the corresponding sequence to sequence transformation matrix was a scalar multiple of the same matrix or of the same matrix without its first row or column. In the present paper the most general matrices are found which have this property, in each case as a solution of a partial difference equation.

In matrix notation, if $A \equiv (a_{nk})$ $(n, k = 0, 1, \cdots)$ is a series to series transformation matrix, the corresponding series to sequence transformation matrix G is given by $g_{nk} = a_{0k} + a_{1k} + \cdots + a_{nk}$, and the corresponding sequence to sequence transformation matrix F by $f_{nk} = g_{nk} - g_{n\,k+1}$, [ST (2.4), (2.15)].

If E, R and S are the matrices

$$(1.1) \qquad E \equiv \begin{matrix} 0, 0, 0, \cdots \\ 1, 0, 0, \cdots \\ 0, 1, 0, \cdots \\ 0, 0, 1, \cdots \\ \cdots\ \cdots \end{matrix}, \quad R \equiv \begin{matrix} 1, 1, 0, 0, \cdots \\ 1, 1, 0, 0, \cdots \\ 1, 1, 1, 0, \cdots \\ 1, 1, 1, 1, \cdots \\ \cdots\ \cdots \end{matrix}, \quad S \equiv \begin{matrix} 1, & 0, & 0, 0, \cdots \\ -1, & 1, & 0, 0, \cdots \\ 0, & -1, & 1, 0, \cdots \\ 0, & 0, & -1, 1, \cdots \\ \cdot & \cdot & \cdot\ \cdot \end{matrix};$$

so that

$$(1.2) \qquad R = I + E + E^2 + \cdots, \qquad S = I - E, \qquad RS = SR = I,$$

the correspondence between the matrices A and F can be expressed in the form

$$(1.3) \qquad F = (RA)S = R(AS) = RAS,$$

since each element of the product matrix is the sum of a finite number of elements.

* Received August 29, 1949.

[1] "Series to series transformations and analytic continuation by matrix methods," *American Journal of Mathematics*, vol. 71 (1949), pp. 541-562. This paper will be referred to as [ST].

If E' is the transpose of E, the matrix product AE is the matrix A without its first column, and the product $E'A$ is the matrix A without its first row. In what follows we shall find all solutions of the matrix equations

$$RXS = px, \qquad RXS = pXE, \qquad RXS = pE'X, \qquad RXS = pE'XE,$$

where p is an arbitrary fixed number, and R, S, E have the meanings given above. In every case the solution is a matrix $A(p)$ depending on the number p multiplied by a matrix, the elements of which are arbitrary to a certain degree. The matrices $A(p)$ are in the various cases equal or simply related to the series to series transformation matrices of the Taylor series and Laurent series continuation and of the Euler summation matrix, [ST Sections **3**, **4**, **5**].

2. The equation $RXS = pX$. Since R and S are row-finite, we have for every X $S(RXS) = (SR)(XS) = XS$, and $R(SX) = (RS)X = X$, hence equation **2** implies and is implied by

$$XS = pSX. \tag{2.1}$$

It follows from (2.1) that the elements x_{nk} satisfy

$$x_{n\,k+1} = (1-p)x_{nk} + px_{n-1\,k} \qquad k = 0, 1, \cdots, \; n = 1, 2, \cdots, \tag{2.2}$$

$$x_{0\,k+1} = (1-p)x_{0k} \qquad k = 0, 1, \cdots. \tag{2.3}$$

Equation (2.2) is the partial difference equation to be solved, and (2.3) represents incomplete boundary conditions leaving the elements x_{n0} of the first column arbitrary. Taking $x_{00} = 1$, and all other arbitrary elements zero, we obtain step by step all elements of a matrix A which we regard as the *fundamental solution*

$$a_{nk} = \binom{k}{n} p^n (1-p)^{k-n} \text{ for } k \geq n; \; a_{nk} = 0 \text{ for } k < n.$$

Taking $x_{N0} = 1$, and all other arbitrary elements zero, where N is a positive integer, the same matrix results with N rows of zeros added. The fundamental solution is the matrix $A(p)$ of the Taylor series continuation [ST (3.14)], and the matrix with N rows of zeros added can be expressed as $E^N A(p)$, where E is the diagonal vector defined in (1.1). Hence giving arbitrary values d_n to the elements of the first column, the other elements can be found from (2.2) and (2.3) step by step, each of the d_n contributing

linearly a matrix $d_nE^nA(p)$. The *general solution* is therefore the sum of all the contributions

$$X = DA(p), \tag{2.4}$$

where

$$D = d_0I + d_1E + d_2E^2 + \cdots \equiv \begin{matrix} d_0, 0, 0, \cdots \\ d_1, d_0, 0, \cdots \\ d_2, d_1, d_0, \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdots \end{matrix} \tag{2.5}$$

The matrix D, being a power series in E, commutes with every power series in E, in particular with R and S. Hence $RDS = RSD = D$, i.e. the corresponding sequence to sequence transformation matrix is the same matrix D. Regarded as a sequence to sequence transformation matrix, it is a "modified Nörlund matrix."[2] It is regular if and only if $\Sigma \mid d_k \mid$ is convergent and $\Sigma d_k = 1$. In this case it is regular both as a series to series or as a sequence to sequence transformation matrix.

The properties of the matrix $A(p)$ have been discussed in [ST **3**]. It is regular if and only if $0 \leq p \leq 1$. Another property follows easily from (2.1): Since (Ap), $A(q)$, S are column-finite, $A(p)A(q)S = qA(p)SA(q) = qp \cdot SA(p)A(q)$, so that $A(p)A(q)$ is *the* fundamental solution of the equation $XS = pqSX$, i.e. $A(p)A(q) = A(pq)$, as proved by direct calculation in [ST 3. V.].

3. The equation $RXS = pXE$. As shown in Section **2**, this equation is equivalent to

$$XS = pSXE, \tag{3.1}$$

which requires $x_{nk} - x_{n\,k+1} = p(-x_{n-1\,k+1} + x_{n\,k+1})$, and $x_{0k} - x_{0\,k+1} = px_{0\,k+1}$ for $k = 0, 1, 2, \cdots$; $n = 1, 2, \cdots$.

Excluding $p = -1$, and putting $1/(1+p) = t$, we obtain

$$x_{n\,k+1} = tx_{nk} + (1-t)x_{n-1\,k+1}, \text{ for } n = 1, 2, \cdots, \tag{3.2}$$

and

$$x_{0\,k+1} = tx_{0k}. \tag{3.3}$$

[2] G. Piranian, "Nörlund transformations with a bounded base," Office of Naval Research, Project M786, 1949.

The elements of the first column can be chosen arbitrarily, and then all other elements are determinate, and can be found step by step. Taking $x_{00}=1$, and all other arbitrary elements zero, we obtain the *fundamental solution*

$$a_{nk}=\binom{k+n-1}{n}(1-t)^n t^k,$$

which is the matrix $A(t)$ of the Laurent series continuation [ST (5. 3)]. The *general solution* is then obtained, as in Section **2**,

$$X=DA(t). \tag{3.4}$$

When $p=-1$, we obtain $x_{nk}=x_{n-1\ k+1}$ and $x_{0k}=0$, which gives the zero matrix, already included in (3. 4).

4. The equation $RXS=pE'X$. This equation is equivalent to

$$XS=pSE'X, \tag{4.1}$$

which requires, $x_{nk}-x_{n\ k+1}=p(x_{n+1\ k}-x_{nk})$, and $x_{0k}-x_{0\ k+1}=px_{1k}$ for $k=0,1,2,\cdots;\ n=1,2,\cdots$.

Excluding $p=0$, and putting $-1/p=q$, we obtain

$$x_{n+1\ k}=(1-q)x_{nk}+qx_{n\ k+1}, \qquad n=1,2,\cdots, \tag{4.2}$$

and

$$x_{1k}=q(x_{0\ k+1}-x_{0k}). \tag{4.3}$$

Here the elements of the first row can be chosen arbitrarily, and the other elements then determined step by step. Taking $a_{0k}=1$, for a fixed k, and all other elements of the first row zero, (4. 3) determines the values in the second row $a_{1k}=-q$, $a_{1\ k-1}=q$, all the other elements zero. The elements in the other rows are then obtained easily by considering separately the values contributed by a_{1k} and $a_{1\ k-1}$. A short calculation shows that the matrix thus obtained can be expressed as the matrix product $B(q)SC_k$, where $B(q)$ is the bordered Euler series to series summation matrix $\mathcal{E}(q)$: [ST(4. 2)]

$$B(q)\equiv\begin{matrix}1, & 0 & , & 0 & , & 0 & ,0,\cdots\\ 0, & q & , & 0 & , & 0 & ,0,\cdots\\ 0, & q(1-q) & , & q^2 & , & 0 & ,0,\cdots\\ 0, & q(1-q)^2, & & 2q^2(1-q), & & q^3 & ,0,\cdots\\ 0, & q(1-q)^3, & & 3q^2(1-q)^2, & & 3q^3(1-q), & q^4,\cdots\\ \cdot & \cdot & & \cdot & & \cdot & \cdot\ \cdot\ \cdot\end{matrix}; \tag{4.4}$$

S is defined in (1. 2), and C_k is the cross-diagonal vector

$$(4.5)\qquad C_k \equiv \begin{matrix} 0, 0, \cdots, 0, 0, 1, 0, 0, \cdots \\ 0, 0, \cdots, 0, 1, 0, 0, 0, \cdots \\ 0, 0, \cdots, 1, 0, 0, 0, 0, \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ , \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0, 1, \cdots, 0, 0, 0, 0, 0, \cdots \\ 1, 0, \cdots, 0, 0, 0, 0, 0, \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ , \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{matrix} \qquad c_{ni} \begin{cases} = 1 \text{ for } n + i = k, \\ = 0 \text{ otherwise.} \end{cases}$$

The *general solution* is therefore

$$(4.6)\qquad X = B(q)SC, \text{ where } C \equiv \begin{matrix} c_0, c_1, c_2, \cdots \\ c_1, c_2, c_3, \cdots \\ c_2, c_3, c_4, \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \end{matrix}$$

$c_0, c_1, \cdots$ being arbitrary numbers. When $p = 0$, we obtain $x_{nk} = x_{n\ k+1}$ for every n and k, which is satisfied by any matrix with all its columns equal. This is not included in (4. 6).

5. The equation $RXS = pE'XE$. This is equivalent to

$$(5.1)\qquad XS = pSE'XE,$$

which requires, $x_{nk} - x_{n\ k+1} = p(x_{n+1\ k+1} - x_{n\ k+1})$, and $x_{0k} - x_{0\ k+1} = px_{1\ k+1}$ for $k = 0, 1, 2, \cdots$; $n = 1, 2, \cdots$.

Excluding $p = 0$, and putting $1/p = q$, we obtain

$$(5.2)\qquad x_{n+1\ k+1} = (1 - q)x_{n\ k+1} + qx_{nk}, \qquad n = 1, 2, \cdots,$$

and

$$(5.3)\qquad x_{1\ k+1} = q(x_{0k} - x_{0\ k+1}).$$

These equations leave the elements of the first row and first column arbitrary. Taking $x_{00} = 1$, all the other arbitrary elements zero, we obtain as fundamental solution the matrix $B(q)$ defined in (4. 4). If, for a fixed n, $x_{n0} = 1$, the other arbitrary elements being zero, we obtain, as in Section **2**, the matrix $E^nB(q)$, so that the contribution of all the elements of the first column is the matrix $DB(q)$, D being defined in (2. 5). If the only non-zero arbitrary element is $x_{0k} = 1$, for a fixed $k \geq 1$, we obtain, as in Section **4**,

the matrix $B(q)SE'^{k}$. Hence the contribution of arbitrary elements $x_{01} = h_1$, $x_{02} = h_2, \cdots$ is *a* matrix H:

$$(5.4) \qquad H \equiv h_1 E' + h_2 E'^2 + \cdots \equiv \begin{matrix} 0, h_1, h_2, h_3, \cdots \\ 0, 0\ , h_1, h_2, \cdots \\ 0, 0\ , 0\ , h_1, \cdots \\ 0, 0\ , 0\ , 0\ , \cdots \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdots \end{matrix}.$$

The *general solution*, for $p \neq 0$, is therefore

$$(5.5) \qquad X = DB(q) + B(q)SH.$$

When $p = 0$, we obtain the same matrix as in Section **4**.

In the special case, when we take

$$d_n = q(1-q)^n, \quad n = 0, 1, 2, \cdots, \qquad h_k = 0, \quad k = 1, 2, \cdots,$$

so that $X = DB(q)$, we obtain the matrix $\mathcal{E}(q)$ of the Euler series to series transformation.

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SUMMABILITY METHODS WEAKER THAN CONVERGENCE.*

By J. D. Hill.

Consider the Banach space (c) of all real convergent sequences $x \equiv \{s_k\}$ with $\| x \| \equiv \sup_k | s_k |$. Every linear functional $f(x)$ defined in (c) is of the form [1, p. 65] $f(x) = C \lim_k s_k + \sum_{k=1}^{\infty} C_k s_k$ with $\| f \| = | C | + \sum_{k=1}^{\infty} | C_k |$. A sequence

$$(1) \qquad f_n(x) = C_n \lim_k s_k + \sum_{k=1}^{\infty} C_{nk} s_k \qquad (n = 1, 2, 3, \cdots)$$

of such functionals, in which we shall suppose that *not all of the C_n are zero,* defines a method of summability by means of which a sequence x is said to be summable to the value $f(x)$ if $\lim_n f_n(x)$ exists and equals $f(x)$. We shall refer to such a method as a *W-method.* If $f(x)$ is defined for every x in (c) then W is called *conservative.* It is a trivial extension of a standard theorem for Silverman-Toeplitz methods, and may be proved very simply by operational means [see 1, pp. 90-91], that W will be conservative if and only if the following conditions hold:

$$(2) \qquad \lim_{n \to \infty} C_{nk} \equiv L_k \text{ exists}, \qquad (k = 1, 2, 3, \cdots),$$

$$(3) \qquad \lim_{n \to \infty} \{C_n + \sum_{k=1}^{\infty} C_{nk}\} \equiv L \text{ exists},$$

$$(4) \qquad \sup_n \{| C_n | + \sum_{k=1}^{\infty} | C_{nk} |\} < \infty.$$

On the other hand, if $\lim_n f_n(x)$ exists and equals $\lim_k s_k$ for every x in (c), then W is *regular.* The conditions (2), (3), and (4) with all $L_k = 0$ and $L = 1$ are necessary and sufficient for regularity.

A transformation of the type W differs from the more familiar Silverman-Toeplitz matrix method in that the existence of $\lim_k s_k$ is required. Hence the *convergence-field* of W (i. e., the set of all W-summable sequences) is a subset of (c). Such a method is said to be *not stronger than convergence.* If the convergence-field is a *proper* subset of (c) then the method is *weaker than convergence.* The Cesàro methods (C, α) for $-1 < \alpha < 0$ are well known examples of Silverman-Toeplitz methods having this property [see 3, p. 17, Theorem II].

The following question arises. What properties must be possessed by a

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proper subset G of (c) in order that there exist a W-method whose convergence-field is precisely G? It is first of all necessary that G be linear and Borel measurable [1, p. 18, Theorem 9] and consequently of the first category [1, p. 36, Theorem 1]. Closer to the heart of the matter is the fact that G must be a set of type $F_{\sigma\delta}$ [1, p. 18, Theorem 9], namely, a denumerable product of sets each of which is a denumerable sum of closed sets. It is known [1, p. 235] that if G is not closed then it cannot be a set of type F_σ (a denumerable sum of closed sets). The object of this note is to point out that every linear closed set is a set G. Whether every linear $F_{\sigma\delta}$ is a set G is a more difficult question which we propose to make the subject of a later paper. It will be observed from the proof that the following theorem can be rephrased to hold for sequences of linear functionals defined over any separable Banach space. For the present purposes, however, we prefer the form given.

THEOREM. *Let G be a linear proper subset of (c). Then in order that there exist a W-method whose convergence-field is precisely G it is sufficient (but not necessary) that G be closed.*

Proof. To see first that the condition is not necessary, let $X \equiv \{\delta_k{}^k\}$ and $X_n \equiv \{\delta_k{}^n\}$ for $n = 1, 2, 3, \cdots$, where $\delta_k{}^n$ is the Kronecker symbol. Let E denote the (proper) subset of (c) composed of all finite linear combinations of X and the X_n, and consider the W-transformation

$$f_n(x) = \tfrac{1}{2}\lim_k s_k + (2 \log 2)^{-1} \sum_{k=1}^{n} (-1)^{n-k} s_k/(n - k + 1) \qquad (n = 1, 2, 3, \cdots). \tag{5}$$

The method (5) satisfies conditions (2) and (3) but not (4). Hence the convergence-field G of (5) contains E but is a proper subset of (c). It follows that G is not closed since the closure of E is (c) itself [1, p. 65, §3], and G contains E.

Now let G denote any closed linear proper subset of (c) and G^* its complement with respect to (c). Since (c) is a separable space, its subset G^* is likewise separable [3, p. 31], and we designate by $\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3, \cdots$ a dense enumerable subset of G^*. Inasmuch as G is closed, the distance $d(y)$ from G to y is greater than zero for each y in G^*. In particular we have $d(\mathfrak{Y}_n) > 0$ for all n. Hence [1, p. 57, Lemma] for each $n = 1, 2, 3, \cdots$ there exist linear functionals $F_n(x)$ defined in (c) such that (i) $F_n(\mathfrak{Y}_n) = 1$; (ii) $F_n(x) = 0$ for all x in G; and (iii) $\| F_n \| = 1/d(\mathfrak{Y}_n)$. Let $Q_n(x) \equiv F_n(x)/\| F_n \|$ so that $\| Q_n \| = 1$ $(n = 1, 2, 3, \cdots)$ and let $\{F^*_n\}$ denote the sequence

(6) $\quad Q_1; -Q_1, -Q_2; Q_1, Q_2, Q_3; -Q_1, -Q_2, -Q_3, -Q_4; \cdots$

in which the k-th group, set off by semicolons, contains the first k of the Q_n with the sign of $(-1)^{k-1}$.

It is first of all evident from (ii) that the sequence (6) applied to each x_0 in G yields a sequence composed wholly of zeros. We show next that the sequence (6) applied to each y_0 in G^* yields a divergent sequence bounded by $\| y_0 \|$. The boundedness follows directly from the fact that $\| Q_n \| = 1$ for all n. To establish the divergence we recall that $d(y_0) > 0$ and hence that we can fix a point $\mathfrak{Y}_p$ of the dense enumerable subset of G^* so that $\| y_0 - \mathfrak{Y}_p \| < \frac{1}{3}d(y_0)$. It is immediate then that $d(\mathfrak{Y}_p) \geqq \frac{2}{3}d(y_0)$. Consider the equation $F^*_n(y_0) = F^*_n(y_0 - \mathfrak{Y}_p) + F^*_n(\mathfrak{Y}_p)$. From (6), (i), (iii), and the definition of $Q_n(x)$, there exists a sequence of indices $n_k \uparrow \infty$ for which we have $F^*_{n_k}(\mathfrak{Y}_p) = (-1)^k Q_p(\mathfrak{Y}_p) = (-1)^k d(\mathfrak{Y}_p)$; consequently $F^*_{n_k}(y_0) = F^*_{n_k}(y_0 - \mathfrak{Y}_p) + (-1)^k d(\mathfrak{Y}_p)$. The first term on the right does not exceed $\frac{1}{3}d(y_0)$ in absolute value while the second alternates in sign and is never numerically less than $\frac{2}{3}d(y_0)$. This establishes the divergence of the sequence $\{F^*_n(y_0)\}$ for every y_0 in G^*.

Suppose now that $F^*_n(x) = C^*_n \lim_k s_k + \sum_{k=1}^{\infty} C^*_{nk} s_k$ and that $f_n(x)$, given by (1), defines an arbitrary conservative (or regular) W-method such that not all of the numbers $C^*_n + C_n$ are zero. We set

(7) $\quad g_n(x) = F^*_n(x) + f_n(x)$ for x in (c), $\qquad (n = 1, 2, 3, \cdots)$.

It is apparent from what precedes that (7) defines a method W^* which is effective (or regularly effective) in G and definitely ineffective in G^*. This completes the proof.

It may be observed incidentally that the norm condition (4) is satisfied by the method (7). However, there exist W-methods having the required property and for which (4) is not satisfied; for example, the method defined by $h_n(x) \equiv \lambda_n F^*_n(x) + f_n(x)$ where $\lambda_n \to \infty$.

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[3] M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge, 1939.

ERRATA.

W. L. Chow, "On compact complex analytic varieties," this JOURNAL, vol. 71, pp. 893-914.

Page 894, line 8 from the bottom, "The set W_r..." should read "The set $\tilde{W}_r$..."

Page 894, line 5 from the bottom, "The set $W_r - W_r$ of..." should read "The set $W_r - W_r$ of..."

Page 895, line 8, "analytic elements..." should read "regular analytic elements..."

Philip Hartman and Aurel Wintner, "On linear difference equations of second order," this JOURNAL, vol. 72, pp. 124-128.

Page 124, in formula (3), replace the second sign "$>$" by "$<$."

Page 125, line 14, replace "non-decreasing" by "non-increasing."

Page 125, in the first formula line, replace "$2p_k$" by "2."

W. L. Chow, "Algebraic systems of positive cycles in an algebraic variety," this JOURNAL, vol. 72, pp. 247-283.

Page 251, line 18, "Let K be..." should read "let $\check{K}$ be..."

Page 251, line 20, "... over K is called the extension U/K of U/K over K." should read "... over $\check{K}$ is called the extension $U/\check{K}$ of U/K over $\check{K}$."

Page 251, line 25, "... an extension W/K over..." should read "... an extension $W/\check{K}$ over..."

Page 281, line 16, "... in this case the $\mathfrak{m}$-adic ring..." should read "... in this case the $(L \times \mathfrak{o}^*)\mathfrak{m}$-adic ring..."